Problem 1 (multiple choice)

1. Assume that $f \in C_0^\infty(\mathbb{R}^n)$. Which PDE does u solve, if
$$u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{2\hat{f}\xi}{6+2|\xi|^2} e^{ix\cdot\xi}\, d\xi?$$
   (a) $-3\Delta u + u = f$
   (b) $6(-\Delta u + u) = 2f$
   (c) $\Delta u + 3u = f$
   (d) $-6\Delta u + 2u = 2f$
   (e) $-\Delta u + 3u = f$

2. Let us define
$$P_y(x) = P(x,y) = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \frac{y}{(|x|^2+y^2)^{(n+1)/2}}, \quad x \in \mathbb{R}^n,\ y > 0.$$
   Which claims are correct:
   (Grading: Correct choice: $+1$, incorrect choice: -1, no choice: 0. Total points for this question cannot be negative. The points are scaled so that the maximum for this question is 2 points.)
   Select one or more:
   (a) $P_y(x) = y^{-n} P_1(x/y)$ for every $y > 0$
   (b) $P_1(x) = y^{-n} P_y(x/y)$ for every $y > 0$
   (c) $\int_{\mathbb{R}^n} P_y(x)\, dx = 1$ for every $y > 0$
   (d) $P_z * P_y = P_{y+z}$ for every $y, z > 0$
   (e) $\int_0^\infty P_y(x)\, dx = 1$ for every $x \in \mathbb{R}^n$

3. Suppose that $f \in C^\infty(\mathbb{R}^2)$. We want to solve the partial differential equation $u_x - u_{yy} = 0$ in $\Omega = \{(x,y) \in \mathbb{R}^2 \ :\ 0 < x < 1,\ 0 < y < 1\}$.
   In which sets we have to require the boundary condition $u = f$ to be satisfied, if we want to have a well posed problem?
   [You will get points for choosing correct parts of the boundary and loose points for choosing wrong parts of the boundary. Total points cannot be negative.]
   Select one or more:
   (a) $\{(x,y) \in \mathbb{R}^2 \ :\ x = 1 \text{ and } 0 < y < 1\}$
   (b) $\{(x,y) \in \mathbb{R}^2 \ :\ x = 0 \text{ and } 0 \le y \le 1\}$
   (c) $\{(x,y) \in \mathbb{R}^2 \ :\ 0 < x < 1 \text{ and } y = 0\}$
   (d) $\{(x,y) \in \mathbb{R}^2 \ :\ 0 < x < 1 \text{ and } y = 1\}$

4. Which of the following claims are true?
   Grading:
   - Answer not chosen: 0
   - Correct choice: +1
   - Wrong choice: −1
   - Points are scaled so that the maximum points for this exercise are 2.
   - Total points cannot be negative.

   Select one or more:

   (a) Derivative and convolution become multiplication on Fourier side.
   (b) The problem $\Delta u = 0$ in $\mathbb{R}^{n+1}_+$, $u = 0$ on $\partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n$ has a unique solution.
   (c) There exists a function u that satisfies $\Delta u(x) = 0$ for every $x \in \mathbb{R}^n$, $u = 1$ on $\partial B(0,1)$ and $u(\bar{0}) = 0$. Here $\bar{0} = (0,0,\dots,0)$ denotes the origin.)
   (d) $\widehat{((f*g)*h)}(\xi) = \hat{f}(\xi)\hat{g}(\xi)\hat{h}(\xi)$

5. What are the six functions listed below? Choose the alternative that best describes the function.

   (i)
$$u(x) = \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n}\, dS(y)$$

   (ii)
$$u(x) = \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}}, \quad x \neq 0,$$

   where $\alpha(n)$ is the volume of the unit ball in $\mathbb{R}^n$

   (iii)
$$u(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}, \quad t > 0$$

   (iv)
$$u(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\hat{g}(\xi)\cos(|\xi|t) + \hat{h}(\xi)\frac{\sin(|\xi|t)}{|\xi|} \right) e^{ix\cdot\xi}\, d\xi$$

   (v)
$$\widehat{u}(\xi, y) = \hat{g}(\xi) e^{-|\xi| y}, \quad y > 0$$

   (vi)
$$u(x,t) = \frac{1}{|\partial B(x,t)|} \int_{\partial B(x,t)} (th(y) + g(y) + \nabla g(y)\cdot(y-x))\, dS(y)$$

   (a) The solution of the Dirichlet problem for the Laplace equation in a ball
   (b) The fundamental solution of the Laplace equation when the dimension is at least three
   (c) The heat kernel in the upper half-space
   (d) The solution on the Fourier side of the Cauchy problem for the wave equation

(e) The solution on the Fourier side of the Dirichlet problem for the Laplace equation in the upper half-space

(f) Kirchhoff's formula for the solution of the Cauchy problem for the three-dimensional wave equation

6. We consider solving the Dirichlet problem

$$\begin{cases} \Delta u(x,y) = 0, & (x,y) \in B(0,1), \\ u(x,y) = g(x,y), & (x,y) \in \partial B(0,1), \end{cases}$$

in the unit disc $B(0,1) = \{(x,y) \in \mathbb{R}^2 \ : \ x^2 + y^2 < 1\}$ using the separation of variables technique. Which of the following claims are true:

[Grading: you get $+1$ points for correct answers, and -1 points for wrong answers. The points are scaled so that the maximum is 3 points. Total points cannot be negative.]

(a) We search special solutions in the form $A(x)B(y)$.

(b) We search special solutions in the form $A(\theta)B(r)$, where (r,θ) is the representation of the point in plane using polar coordinates.

(c) The Laplace equation will be reduced into two ordinary differential equations.

(d) All obtained special solutions are useful in solving the problem.

(e) The solution to the original problem is obtained as linear combination of the special solutions.

(f) The correct coefficients for the special solutions are obtained from the Fourier coefficients of the boundary values.

Solutions

1. (e)
2. (a), (c), (d)
3. (b), (c), (d)
4. (a), (d)
5. (i)-(a), (ii)-(b), (iii)-(c), (iv)-(d), (v)-(e)
6. (b), (c), (e), (f)