ELEC-C9420 Introduction to Quantum Technology, Fall 21
Midterm exam 2, part B, 16.12.2021
teacher: Matti Raasakka

Problem B1

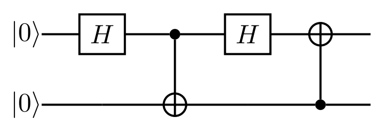
Consider a quantum particle in a 1D harmonic potential $U(x) = \frac{1}{2}kx^2$. The particle has energy eigenstates $|n\rangle$, $n = 0, 1, 2, \ldots$, with corresponding energy eigenvalues $E = \hbar\omega(n + \frac{1}{2})$, where $\omega = \sqrt{k/m}$ is the characteristic angular frequency of the oscillator. At time $t = 0$ the particle is in the state

$$|\phi(0)\rangle = \frac{1}{\sqrt{3}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{6}}|2\rangle \,.$$

a) What is the probability to detect the system in its ground state at $t = 0$? (2p)
b) Compute the expectation value of energy for the particle at $t = 0$. (2p)
c) Find the state vector $|\phi(T)\rangle$ after the system is let to evolve for time T. (2p)
d) Let's imagine the energy of the particle is measured at time T, and found to be $\frac{3}{2}\hbar\omega$. The system is then let to evolve for time T'. What is the probability to find the particle in its ground state at time $t = T + T'$ in this case? (2p)

Problem B2

Consider the following quantum circuit.



a) What is the output state of the circuit? (2p)
b) Are the qubits entangled in the output state? (2p)
c) What are the probablities to measure different qubit values in the output state? (2p)
d) Find the expectation value for the sum of the qubit values in the output state. (2p)

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Problem B1

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Model solution

a) The ground state of the particle is the lowest energy state $|0\rangle$. (1p) The probability to detect the particle in this state according Born rule is

$$|\langle 0|\phi(0)\rangle|^2 = \frac{1}{3}\,. \quad \text{(1p)}$$

b) The expectation value of the energy is

$$\langle\phi(0)|\hat{H}|\phi(0)\rangle = \frac{1}{3}\frac{\hbar\omega}{2} + \frac{1}{2}\frac{3\hbar\omega}{2} + \frac{1}{6}\frac{5\hbar\omega}{2} = \frac{4}{3}\hbar\omega\,. \quad \text{(2p)}$$

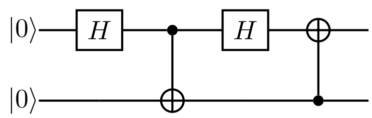
c) Since the time-evolution operator U_t is linear, each one of the energy eigenstates evolves independently in time as $U_t|n\rangle = e^{-iE_n t/\hbar}|n\rangle$, where $E_n = \hbar\omega(n + \frac{1}{2})$. (1p) Accordingly, for the state of the particle at time $t = T$ we get

$$|\phi(T)\rangle = \frac{1}{\sqrt{3}}e^{-i\omega T/2}|0\rangle - \frac{1}{\sqrt{2}}e^{-3i\omega T/2}|1\rangle + \frac{1}{\sqrt{6}}e^{-5i\omega T/2}|2\rangle\,. \quad \text{(1p)}$$

d) Detecting the energy of the particle to be $\frac{3}{2}\hbar\omega$ collapses the state of the particle to the energy eigenstate $|1\rangle$. (1p) Thus, after the measurement at time $t = T$ the state of the particle evolves as $|\phi(t)\rangle = e^{-3i\omega(t-T)/2}|1\rangle$. The probability to measure the particle in its ground state at time $T + T'$ is therefore $|\langle 0|\phi(T + T')\rangle|^2 = 0$ for any $T' > 0$. (1p)

Problem B2

Consider the following quantum circuit.



a) What is the output state of the circuit? (2p)
b) Are the qubits entangled in the output state? (2p)
c) What are the probablities to measure different qubit values in the output state? (2p)
d) Find the expectation value for the sum of the qubit values in the output state. (2p)

Model solution

a) After the first Hadamard gate the state is

$$|\phi_1\rangle = (H \otimes I)|00\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)\,. \quad (0.5\text{p})$$

After the first cNOT gate the state is

$$|\phi_2\rangle = CNOT_{1,2}\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\,. \quad (0.5\text{p})$$

After the second Hadamard gate the state is

$$|\phi_3\rangle = (H \otimes I)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{2}((|0\rangle + |1\rangle)|0\rangle + (|0\rangle - |1\rangle)|1\rangle) = \frac{1}{2}(|00\rangle + |10\rangle + |01\rangle - |11\rangle)\,. \quad (0.5\text{p})$$

After the second cNOT gate the output state is

$$|\phi_4\rangle = CNOT_{2,1}\frac{1}{2}(|00\rangle + |10\rangle + |01\rangle - |11\rangle) = \frac{1}{2}(|00\rangle + |10\rangle + |11\rangle - |01\rangle)\,. \quad (0.5\text{p})$$

b) Let's try to express the state as a product state

$$|\phi\rangle|\psi\rangle = \frac{1}{\sqrt{2}}(\phi_0|0\rangle + \phi_1|1\rangle)\frac{1}{\sqrt{2}}(\psi_0|0\rangle + \psi_1|1\rangle) = \frac{1}{2}(\phi_0\psi_0|00\rangle + \phi_0\psi_1|01\rangle + \phi_1\psi_0|10\rangle + \phi_1\psi_1|11\rangle)\,.$$

(The extra $1/\sqrt{2}$ factors are introduced here for later convenience, but are not necessary. They only change the normalization of the coefficients ϕ_i, ψ_i, but don't affect the product structure of the state.) From here we get four equations for the coefficients

$$\begin{cases} \phi_0\psi_0 = 1 \\ \phi_0\psi_1 = -1 \\ \phi_1\psi_0 = 1 \\ \phi_1\psi_1 = 1 \end{cases}.$$

Now, dividing the second equation by the first equation we get $\psi_1/\psi_0 = -1$. On the other hand, dividing the last equation by the second last we get $\psi_1/\psi_0 = 1$, which is a contradiction. Therefore, the equations cannot be solved, and the output state cannot be expressed as a product state. Thus, the qubits are entangled. (2p)

NOTE: An alternative way to check for entanglement is to compute the reduced state of one of the qubits, and show that it is a mixed state.

c) The output state is an equal superposition over all the basis states, so all pairs of qubit values have probability 1/4. (2p)

d) We could use the general formula $\langle S\rangle = \langle\phi_4|\hat{S}|\phi_4\rangle$ to compute the expectation value of the observable $\hat{S} = \hat{b} \otimes I + I \otimes \hat{b}$, which corresponds to the sum of the two qubit values. (Here, $\hat{b}$ is the observable corresponding to the value of a single qubit.) However, since the basis states are orthonormal, thus representing mutually exclusive possibilities, we can also just apply the formula from classical probability calculus for the expectation value of a statistical variable X: $\langle X\rangle = \sum_x xP(x)$, where we sum over all the possible values x of the variable X, and $P(x)$ is the probability of the value x. We get for the sum of the qubit values

$$\frac{1}{4}\cdot(0+0) + \frac{1}{4}\cdot(0+1) + \frac{1}{4}\cdot(1+0) + \frac{1}{4}\cdot(1+1) = 1\,. \quad (2\text{p})$$