MS-E1461 Hilbert spaces (Aalto University, Turunen)
Examination on Tuesday 21.12.2021, at 9:00-13:00

Points also for good effort! You may use websites and the course material. Yet, the problems must be solved individually. Please indicate the sources you used.

Note: In the following, H is an infinite-dimensional **complex** Hilbert space. with inner product $(u, v) \mapsto \langle u, v\rangle$ and norm $u \mapsto \|u\| = \langle u, u\rangle^{1/2}$.

1. Let v_1, v_2 be two linearly independent vectors in a Hilbert space.

   (a) Find an orthonormal basis for vector subspace $Z = \mathrm{span}(\{v_1, v_2\})$.

   (b) Find the formulas for the orthogonal projection P onto Z, and for the orthogonal projection Q onto the orthogonal complement of Z.

   (c) Sketch also a picture (or pictures) how these vectors and projections look like in case of a 3-dimensional real Hilbert space.

2. In this problem, Hilbert space H is Lebesgue space $L^2(\mathbb{R}^+)$.

   (a) Show that $B(u, v) = \langle Lu, Lv\rangle$, where

   $$B(u,v) := \int_0^\infty \int_0^\infty \frac{u(x)\, v(y)^*}{x+y}\, \mathrm{d}x\, \mathrm{d}y \quad \text{and} \quad Lv(x) := \int_0^\infty \mathrm{e}^{-xy}\, v(y)\, \mathrm{d}y.$$

   (b) Show that here B is a bounded Hermitian form for which $B(u, u) \geq 0$.

   (Hint: Apply Hilbert's Inequality.)

3. The *exponential* of $A \in \mathscr{B}(H)$ is defined by $\exp(A) := \sum_{k=0}^{\infty} \frac{A^k}{k!}$.

   (a) Show that $\exp(A)\exp(B) = \exp(A+B)$ if $AB = BA$. Show also that $\frac{\mathrm{d}}{\mathrm{d}t}\exp(tA) = A\exp(tA)$, where derivative with respect to variable $t \in \mathbb{R}$ is defined as the limit of the difference quotient (as usual).

   (b) Show that $A^* = A$ if and only if $\exp(\mathrm{i}tA)$ is unitary for all $t \in \mathbb{R}$.

   (Hint: For $U = \exp(\mathrm{i}tA)$, insert $I = U^*U$ to $0 = \frac{\mathrm{d}}{\mathrm{d}t} I\big|_{t=0} = ...$)

4. For $v : \mathbb{Z}^+ \to \mathbb{C}$, define $Kv : \mathbb{Z}^+ \to \mathbb{C}$ by

   $$Kv(x) := (-3)^{-x}\, v(x+3).$$

   (a) Show that $K : H \to H$ is a compact linear operator in $H = \ell^2(\mathbb{Z}^+)$.

   (b) Find the eigenvalues and eigenvectors of $L = K^*K$. Diagonalize L.

   (c) Find the SVD (Singular Value Decomposition) for K.