ELEC-A7200 Signals and systems

Autum 2019, Retake exam

13.01.2020

Note. If you have also taken one or both middle term exams, of the corresponding tasks of this exam and middle term exams into your course grade evaluation is picked the one of which you have earned more points.

Task 1

a) (2p.) Let $x_1(t)$ and $x_2(t)$ be orthonormal energy signals. Solve

$$\langle x_1(t) - 2x_2(t), 3x_1(t)\rangle \quad \text{and} \quad \langle 3x_1(t) - 2x_2(t), x_2(t)\rangle$$

b) (2p.) Let $x_3(t) = \text{tria}(t-1)$, where

$$\text{tria}(t) = \begin{cases} 1+t, & -1 \le t < 0 \\ 1-t, & 0 \le t \le 1 \\ 0, & \text{else.} \end{cases}$$

Present signal $x_4(t) = \frac{dx_3(t)}{dt}$ as a linear combination of rectangular pulses of form $\text{rect}\left(\frac{t-t_0}{T}\right)$, where

$$\text{rect}(t) = \begin{cases} 1, & |t| \le \frac{1}{2} \\ 0, & |t| > \frac{1}{2}. \end{cases}$$

Draw graphs of the signals $x_3(t)$ and $x_4(t)$.

c) (2p.) Solve

$$\int_{-\infty}^{\infty} x_3(t)\, \delta(t-1)\, dt,$$

where $\delta(t)$ is Dirac's delta function.

d) (4p.) Let $x_5(t) = e^{-2t}u(t)$, where $u(t)$ is unit step function:

$$u(t) = \begin{cases} 1, & t \ge 0 \\ 0, & t < 0. \end{cases}$$

Solve convolution of $x_5(t)$ with itself:

$$y(t) = \int_{-\infty}^{\infty} x_5(\tau)x_5(t-\tau)\, d\tau.$$

Hint: Draw a graph.

Task 2

Consider triangular pulse

$$x(t) = \begin{cases} 1-|t|, & |t| \le 1 \\ 0, & |t| > 1 \end{cases}$$

a) (3p.) Calculate signal energy of $x(t)$.

b) (2p.) Find Fourier transform $Y(f)$ of the signal $y(t) = x(\alpha t)$.

c) (2p.) Find Fourier transform $Y(f)$ of the signal $y(t) = x(t-\beta)$.

d) (3p.) Find Fourier transform $Y(f)$ of the signal $y(t) = x(\alpha(t-\beta))$.

Task 3

a) (2p.) Find Fourier transform $X(f)$ of the signal $x(t) = 2\cos(2\pi f_c t)$.

b) (2p.) Find Fourier transform $P(f)$ of the signal $p(t) = \text{rect}\left(\frac{t}{T}\right)$, when $T > 0$.

c) (4p.) Find Fourier transform $Y(f)$ and energy spectral density $|Y(f)|^2$ of the signal $y(t) = x(t)p(t)$

d) (2p.) Find energy of the pulse $y(t)$ when $f_c = 1$ and $T = 1$.

Task 4

Pulse

$$x(t) = \begin{cases} t, & 0 < t \le \frac{3}{2} \\ 3-t, & \frac{3}{2} < t \le 3 \\ 0, & \text{else,} \end{cases}$$

is sampled with sampling period $T_s = 1$ on interval $[0, 3]$.

a) (1p.) Find sampling sequence $\{x(n)\} = \{x(0), x(1), x(2), x(3)\}$.

b) (3p.) Find Discrete Fourier Transform (DFT) $\{X(k)\}$ of $\{x(n)\}$.

c) (2p.) What frequencies do indexes $k = 0, 1, 2, 3$ in $\{X(k)\}$ correspond to?

d) (2p.) $\{x(n)\}$ is zero padded with eight zeros. How does this affect the frequency resolution of DFT?

e) (2p.) Explain why aliasing occurs when pulse $x(t)$ is sampled.

Task 5

Consider LTI-system whose input $x(t)$ and response $y(t)$ satisfy differential equation

$$2\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = x.$$

a) (5p.) Find frequency response $H(f)$ of the system.

b) (5p.) Calculate system's amplitude response $A(f) = |H(f)|$ and phase response $\phi(f) = -\arg\{H(f)\}$ for frequencies 0.1 Hz and 10 Hz.

Task 6

Let autocorrelation function of a random signal $\tilde{\mathbf{y}}$ be

$$r_{\tilde{y}\tilde{y}}(\tau) = E\{\tilde{y}(t)\tilde{y}^*(t-\tau)\} = \exp(-\frac{2|\tau|}{\alpha})$$

a) (2p.) Find average power $E\{|\tilde{y}(t)|^2\}$ of $\tilde{\mathbf{y}}$.

b) (3p.) Find power spectrum $S_{yy}(f)$ of the signal.

Let power spectrum of a white noise be $S_{zz}(f) = N_0/2$

c) (2p.) Bandwidth of some signal is B. Find noise power of the noise when it is bandlimited on the frequence band of that signal.

d) (3p.) Find frequency response $H(f)$ of a *stable* filter so that $S_{yy}(f) = |H(f)|^2 S_{zz}(f)$.

Theorems of the fourier transform	**Function**	**Transform**
Linearity	$ax(t)+by(t)$	$aX(f)+bY(f)$
Time delay or time shift	$x(t-a)$	$X(f)e^{-j2\pi fa}$
Scale change	$x(at)$	$\frac{1}{\lvert a\rvert}X(\frac{f}{a})$
Conjugation	$x^*(t)$	$X^*(-f)$
Duality	$X(t)$	$x(-f)$
Frequency shift	$x(t)e^{j2\pi at}$	$X(f-a)$
Linear modulation	$x(t)\cos(2\pi at+b)$	$\frac{e^{jb}X(f-a)+e^{-jb}X(f+a)}{2}$
Differentiation	$\frac{d^n x(t)}{dt^n}$	$(j2\pi f)^n X(f)$
Integration	$\int_{-\infty}^{t} x(u)du$	$\frac{X(f)}{j2\pi f}$
Convolution	$x(t)\otimes y(t)$	$X(f)Y(f)$
Multiplication	$x(t)y(t)$	$X(f)\otimes Y(f)$
Multiplication by t^n	$t^n x(t)$	$-\frac{1}{j2\pi}\frac{d^n X(f)}{df^n}$

Fourier transforms	**Function**	**Transform**
Rectangular pulse	$\mathrm{rect}(t/a)$	$a\cdot\mathrm{sinc}(af)$
Triangular pulse	$\mathrm{tria}(t/a)$	$a\cdot\mathrm{sinc}^2(af)$
Gaussian pulse	$e^{-\pi(\frac{t}{a})^2}$	$a\cdot e^{-\pi(af)^2}$
One sided exponential pulse	$e^{-t/a}u(t)$	$\frac{a}{1+j2\pi fa}$
Two sided exponential pulse	$e^{-\lvert t\rvert/a}$	$\frac{2a}{1+(2\pi fa)^2}$
Sinc pulse	$\mathrm{sinc}(at)$	$\frac{1}{a}\mathrm{rect}(f/a)$
Constant	a	$a\cdot\delta(f)$
Phasor	$e^{j(2\pi at+b)}$	$e^{jb}\delta(f-a)$
Cosine wave	$\cos(2\pi at+b)$	$\frac{e^{jb}\delta(f-a)+e^{-jb}\delta(f+a)}{2}$
Delayed impulse	$\delta(t-a)$	$e^{-j2\pi fa}$
Step	$u(t)$	$\frac{\delta(f)}{2}+\frac{1}{j2\pi f}$

$$T_0=\frac{1}{f_0}=\frac{2\pi}{\omega_0}$$

$$e^{j\phi}=\cos(\phi)+j\sin(\phi)$$

$$\sin^2\phi+\cos^2\phi=1$$

$$\cos(\phi)=\sin(\phi+\pi/2)$$

$$\sin(\phi)=\cos(\phi-\pi/2)$$

$$\cos^2(\phi)=\frac{1}{2}\Big[1+\cos(2\phi)\Big]$$

$$\sin^2(\phi)=\frac{1}{2}\Big[1-\sin(2\phi)\Big]$$

$$\cos^3(\phi)=\frac{1}{4}\Big[3\cos(\phi)+\cos(3\phi)\Big]$$

$$\sin^3(\phi)=\frac{1}{4}\Big[3\sin(\phi)-\sin(3\phi)\Big]$$

$$\sin(\alpha)\cos(\beta)=\frac{\sin(\alpha-\beta)+\sin(\alpha+\beta)}{2}$$

$$\sin(\alpha)\sin(\beta)=\frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}$$

$$\cos(\alpha)\cos(\beta)=\frac{\cos(\alpha-\beta)+\cos(\alpha+\beta)}{2}$$

$$x(t)=\sum_{k=-\infty}^{\infty}x_k e^{j2\pi kf_0t}=\frac{\alpha_0}{2}+\sum_{k=1}^{\infty}\Big[\alpha_k\cos(2\pi kf_0t)+\beta_k\sin(2\pi kf_0t)\Big]$$

$$x_k=\frac{1}{T_0}\int_{T_0}x(t)e^{-j2\pi kf_0t}dt$$

$$\alpha_k=2\cdot\mathrm{Re}\{x_k\},\quad \text{when } x(t)\in\mathbb{R}$$

$$\beta_k=-2\cdot\mathrm{Im}\{x_k\},\quad \text{when } x(t)\in\mathbb{R}$$

$$X(f)=\mathcal{F}\Big\{x(t)\Big\}=\int_{-\infty}^{\infty}x(t)e^{-j2\pi ft}dt$$

$$x(t)=\mathcal{F}^{-1}\Big\{X(f)\Big\}=\int_{-\infty}^{\infty}X(f)e^{j2\pi ft}df$$

$$X_k=\sum_{n=0}^{N-1}x_n e^{-j2\pi kn/N}$$

$$x_k=\frac{1}{N}\sum_{n=0}^{N-1}X_n e^{j2\pi kn/N}$$

$$f_0=\frac{1}{N\cdot T_s}=\frac{f_s}{N}$$

$$s=\sigma+j\omega=\sigma+j2\pi f$$

$$X(s)=\mathcal{L}\Big\{x(t)\Big\}=\int_{0}^{\infty}x(t)\cdot e^{-st}dt$$

$$d_n=\frac{u_n}{u_1}$$

$$d_{\mathrm{tot}}=\sqrt{\sum_{n=2}^{\infty}d_n^2}$$