Instructions: Answer as many questions as possible. For the exam mark, problems (labelled with numbers) are weighted proportionally to the amount of subquestions (labelled with letters) that they contain. Each subquestion has equal weight.

1. Let
$$B = \begin{bmatrix} a & b & b \\ b & a & b \\ b & b & a \end{bmatrix},$$
where $a, b > 0 \in \mathbb{R}$ are real parameters.

   (a) Explain why the following statement is true: the linear system $B\mathbf{x} = \mathbf{b}$ has a solution for every right hand side $\mathbf{b} \in \mathbb{R}^3$ except in the following two cases: $a = b$ or $a = -2b$.

   (b) Give the general definition of $N(M)$, the null space of a matrix M. Then, compute a basis for $N(B)$ in the special case $a = -2, b = 1$ and in the special case $a = b = 5$.

   (c) Give the definition of a symmetric positive definite matrix. In the special case $b = 1$, for which values of a is B positive definite?

2. Let $C = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$.

   (a) Prove that
$$e^{tC} = \begin{bmatrix} 1+2t & t \\ -4t & 1-2t \end{bmatrix}.$$
   Hint: Show first that $C^2 = 0$. What does it imply?

   (b) Calculate e^{I+C}, where I is the 2×2 identity matrix. *Hint: Using again that $C^2 = 0$, what can you say about the powers $(I + C)^k$?*

3. Let
$$A = \begin{bmatrix} 1 & 1 \\ 1 & -5 \\ 1 & -5 \\ 1 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 6 \\ 6 \\ -1 \end{bmatrix}.$$

   (a) Using the Gram-Schmidt algorithm or otherwise, compute a QR decomposition of the matrix A.

   (b) Compute the nullspace of the matrix A and say explain how to use the result to determine the number of solutions of the least square problem
$$\min_{\mathbf{x} \in \mathbb{R}^2} \|A\mathbf{x} - \mathbf{b}\|_2.$$
   Then, using part (a) or otherwise, compute all the solutions to such least square problem.

4. The sequence of Fibonacci numbers is defined as $f_0 = 0$, $f_1 = 1$, and $f_{n+1} = f_n + f_{n-1}$ for all $n \geq 1$.

   (a) Write down a matrix $F \in \mathbb{R}^{2\times 2}$ that satisfies

   $$F \begin{bmatrix} f_n \\ f_{n-1} \end{bmatrix} = \begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix} \qquad \forall\, n \geq 1.$$

   Then, prove that for all $n \geq 0$ it holds

   $$\begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix} = F^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

   (b) Prove that the eigenvalues of the matrix F obtained in item (a) are ϕ and $-1/\phi$, where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. Then, compute also a pair of orthonormal eigenvectors and write a corresponding decomposition $F = Q\Lambda Q^T$ where Q is orthogonal and Λ is diagonal.

   (c) Using items (a) and (b), prove that

   $$f_n = \frac{\phi^n - (-1/\phi)^n}{\phi + 1/\phi}$$

   where f_n is the n-th Fibonacci number and ϕ is the golden ratio.