First course in probability and statistics
MS-A0501, MS-A0502, MS-A0503, MS-A0504, MS-A0509 — Exam
Department of Mathematics and Systems Analysis — 19.2.2020
Aalto University — J Kohonen / G Metsalo

Instructions: Answer in English. Write briefly and clearly, but give reasons for your answers. A number only as an answer does not yield points. The exam has 4 problems, each worth 0-6 points. Each answer sheet must contain:

- Course name and code
- LAST NAME, FIRST NAMES, STUDENT NUMBER (in block letters)
- Study program
- Date and signature

Allowed equipment: calculator, A4-size note sheet (handwritten, text only on one side, own name in upper right corner, no need to return)

P1 An ordinary coin is tossed three times, and the results are denoted with random numbers X_1, X_2, X_3, where 0 means tails and 1 means heads.

(a) List all 8 possible values of the random vector (X_1, X_2, X_3), and their probabilities. **(1p)**

(b) For each possible value of the random vector, express the corresponding values of the random numbers $S = X_1 + X_2$ and $T = X_2 + X_3$. **(1p)**

(c) Give the distributions of S and T as tables. Find the expected values of S and T. **(2p)**

(d) Give the joint distribution of S and T as a table. Based on the table, explain whether S and T are stochastically independent. Calculate also the conditional probabilities $\mathbb{P}(T = 1 \mid S = 2)$ and $\mathbb{P}(T = 2 \mid S = 1)$. **(2p)**

P2 The lecturer has three 6-sided dice. Two of them are fair (results $1, \ldots, 6$ are equally probable) and the third one is loaded so that the result i has probability $i/21$, for $i = 1, \ldots, 6$. The lecturer has chosen one die at random, and rolled it five times, obtaining the result sequence $(6, 6, 2, 4, 6)$. We denote $\Theta = 1$ if the chosen die is fair, and $\Theta = 2$ if it is loaded.

(a) Find the probability of obtaining exactly this result sequence, if the die was fair. **(1p)**

(b) Find the probability of obtaining exactly this result sequence, if the die was loaded. **(1p)**

(c) Find the posterior distribution of Θ. **(3p)**

(d) Using that posterior distribution, calculate the probability of obtaining three sixes, if the chosen die is rolled three more times. **(1p)**

P3 Insects hit the windshield of a car randomly, so that the interval X_i between two consecutive hits is exponentially distributed with an unknown rate parameter of $\lambda > 0$ (insects per minute), that is, with density function

$$f_{X_i|\Lambda}(x_i \mid \lambda) = \lambda e^{-\lambda x_i},$$

if $x_i > 0$. The intervals are independent from each other. Three consecutive intervals between insect hits were $2.2, 0.5$ and 3.0 (minutes).

(a) Find the maximum likelihood estimate for the parameter λ. **(2p)**

(b) If the rate of an exponential distribution is λ, then its mean is $\mu = 1/\lambda$. Find an estimate for λ by solving it from this equation, replacing the unknown parameter μ with the sample mean of the observed intervals. **(1p)**

(c) The rate parameter is treated as a random variable Λ, whose prior distribution is exponential, with the density function

$$f_\Lambda(\lambda) = 2e^{-2\lambda},$$

if $\lambda > 0$. Find the posterior distribution of the rate parameter Λ, and find the point where the posterior density is maximized (posterior mode). **(3p)**

P4 A certain party got 15.0 % of the votes in the last election. A new poll has been conducted, where 1 000 randomly chosen people expressed which party they support. Out of them 180 said they support this party.

(a) Calculate a confidence interval for the current support of this party in the population, using the confidence level 95 %. Use the normal approximation. For the standard deviation, use an estimate that is calculated from the sample proportion 0.18. (Do not use the conservative estimate that is calculated from the proportion 0.5). **(3p)**

(b) At significance level $\alpha = 0.05$, test the null hypothesis that the support of the party is still the same as in the last election, against the alternative hypothesis that the support has changed. Use the normal approximation. Calculate the p-value and express whether the null hypothesis is rejected or not. **(3p)**

Normal distribution table

The table below contains numerical values of the standard normal cumulative distribution function

$$\Phi(x) = F_Z(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,\mathrm{d}t$$

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999