Aalto University
Department of Mathematics and Systems Analysis
MS-C1541 — Metric spaces, 2021/III

Exam KT / T0
25.2.2021
K Kytölä & D Adame-Carrillo

This is the exam sheet for both the course exam (KT) and the final exam (T0) of MS-C1541 Metric spaces. The grading is based on either

- 50% course exam (KT) + 50% exercises (during period III course);
- 100% final exam (T0).

You can attempt both options, and the one leading to the more favorable grade is taken into account.

Depending on the option above, you should solve the following problems:

- **Course exam (KT):** Choose any five of the six problems.
- **Final exam (T0):** Solve all six problems.

(If you solve all problems, the best five are taken into consideration for the course completion option based on course exam + exercises.)

Problems

Problem 1. **(6 pts)**
Consider the following subsets of the Euclidean plane $\mathbb{R}^2$:

$$\begin{aligned} A_1 &= \Big\{ (x,y) \in \mathbb{R}^2 \;\Big|\; -3 \le x \le 3,\ 0 \le y \le 8 \Big\}; \\ A_2 &= \Big\{ (x,y) \in \mathbb{R}^2 \;\Big|\; y = x^3 - x \Big\}; \\ A_3 &= \Big\{ (x,y) \in \mathbb{R}^2 \;\Big|\; 5x^4 + 7y^6 < 11 \Big\}. \end{aligned}$$

(a) Which of the sets $A_1, A_2, A_3 \subset \mathbb{R}^2$ are open?
(b) Which of the sets $A_1, A_2, A_3 \subset \mathbb{R}^2$ are closed?
(c) Which of the sets $A_1, A_2, A_3 \subset \mathbb{R}^2$ are compact?

(Justify your answers concisely.)

Problem 2. **(6 pts)**
Let (X, d_X) and (Y, d_Y) be two metric spaces. Suppose that $f\colon X \to Y$ is a function such that for some $\alpha > 0$ and $M > 0$ we have

$$\mathsf{d}_Y\big(f(x), f(x')\big) \;\le\; M \,\big(\mathsf{d}_X(x,x')\big)^{\alpha} \qquad \text{for all } x, x' \in X.$$

Prove that if $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X, then $\big(f(x_n)\big)_{n\in\mathbb{N}}$ is a Cauchy sequence in Y.

Problem 3. **(6 pts)**
Let (X, d_X) and (Y, d_Y) be two metric spaces. Consider the Cartesian product

$$X \times Y \;=\; \Big\{ (x, y) \;\Big|\; x \in X,\ y \in Y \Big\},$$

consisting of ordered pairs (x, y) with the first component x in X and second component y in Y.

Show directly from the definitions that the formula

$$\mathsf{d}\big((x_1, y_1), (x_2, y_2)\big) \;=\; \mathsf{d}_X(x_1, x_2) + \mathsf{d}_Y(y_1, y_2) \qquad \text{for } (x_1, y_1), (x_2, y_2) \in X \times Y$$

defines a metric on $X \times Y$.

Problem 4. **(6 pts)**

(a) Let X be a compact metric space and $f\colon X \to \mathbb{R}$ a continuous function such that $f(x) > 0$ for all $x \in X$. Prove (using results from the course) that there exists a $c > 0$ such that $f(x) \geq c$ for all $x \in X$.

(b) Give an example of a continuous function $g\colon Y \to \mathbb{R}$ on a non-compact metric space Y such that $g(x) > 0$ for all $x \in Y$ but there does not exist any $c > 0$ such that $g(x) \geq c$ for all $x \in Y$.
(Detailed proofs of continuity and non-compactness are not required here.)

(c) Give an example of a non-continuous function $h\colon Z \to \mathbb{R}$ on a compact metric space Z such that $h(x) > 0$ for all $x \in Z$ but there does not exist any $c > 0$ such that $h(x) \geq c$ for all $x \in Z$.
(Detailed proofs of non-continuity and compactness are not required here.)

Problem 5. **(6 pts)**
Let (X, d_X) and (Y, d_Y) be two metric spaces and $f\colon X \to Y$ a continuous function. Suppose that X is path-connected. Show that then the image $f[X] \subset Y$ is also path-connected.

Problem 6. **(6 pts)**

(a) Prove (using results from the course and exercises) that the series

$$f(x) \;=\; \sum_{n=1}^{\infty} 3^{-n} \sin(4^n x)$$

converges for all $x \in \mathbb{R}$ and defines a continuous function $f\colon \mathbb{R} \to \mathbb{R}$.

(b) Prove that the function $f\colon \mathbb{R} \to \mathbb{R}$ defined in part (a) is not differentiable at 0, i.e., the derivative $f'(0)$ does not exist.

<u>Hint</u>: *First calculate the value $f(0)$. It may be useful to show that at the points $x_j = \pi\, 4^{-j}$ for $j \in \mathbb{N}$, one has $f(x_j) \geq \frac{1}{\sqrt{2}}\, 3^{-j-1}$. To prove non-existence of the derivative $f'(0)$, consider the appropriate difference quotients.*