Aalto University
Department of Mathematics and Systems Analysis
MS-C1541 — Metric spaces, 2021/III

Exam T01 / KT
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This is the exam sheet for both the the final exam (T01) and for the retake of the course exam (KT) of MS-C1541 Metric spaces. The grading is based on either

- 100% final exam (T01);
- 50% course exam (KT) + 50% exercises (during the period III course).

You can attempt both options, and the one leading to the more favorable grade is taken into account.

Depending on the option above, you should solve the following problems:

- **Final exam (T01):** Solve all six problems.
- **Course exam (KT):** Choose any five of the six problems.

(If you solve all problems, the best five are taken into consideration for the course completion option based on course exam + exercises.)

Problems

Problem 1. **(6 pts)**
Consider the functions

$$d_1 \colon \mathbb{R} \times \mathbb{R} \to [0, \infty) \qquad d_1(x, y) = \sqrt{|x - y|} \quad \text{for } x, y \in \mathbb{R},$$
$$d_2 \colon \mathbb{R} \times \mathbb{R} \to [0, \infty) \qquad d_2(x, y) = |x - y|^2 \quad \text{for } x, y \in \mathbb{R}.$$

Which one of these is a metric on the set $\mathbb{R}$ of real numbers? Prove all conditions of a metric for it. For the other one, show concretely that some required property of a metric fails.

Problem 2. **(6 pts)**
Suppose that X and Y are metric spaces and $f \colon X \to Y$ is a continuous function.

(a) Is it true in general that if $A \subset X$ is closed, then also

$$f[A] = \Big\{ y \in Y \;\Big|\; y = f(a) \text{ for some } a \in A \Big\} \subset Y$$

is closed? If yes, justify why; if no, give a counterexample.

(b) Is it true in general that if $B \subset Y$ is closed, then also

$$f^{-1}[B] = \Big\{ x \in X \;\Big|\; f(x) \in B \Big\} \subset X$$

is closed? If yes, justify why; if no, give a counterexample.

Problem 3. **(6 pts)**
Suppose that $u, v \in \mathsf{V}$ are vectors in an inner product space $(\mathsf{V}, \langle\cdot,\cdot\rangle)$ such that the inner products among them are

$$\langle u, u\rangle = \frac{5}{9}, \qquad \langle u, v\rangle = -\frac{10}{21}, \qquad \langle v, v\rangle = \frac{20}{49}.$$

Directly using the defining properties of inner products, show that there exists a constant $\alpha \in \mathbb{R}$ such that

$$u = \alpha\, v,$$

and find the value of α.

Hint: *To get started, consider whether $u - \alpha\, v$ could be the zero vector.*

Problem 4. **(6 pts)**
Let (X, d) be a metric space. Suppose that $K_1, K_2, K_3, \ldots \subset X$ are compact subsets of X. Consider the set

$$A = \left\{ x \in X \;\middle|\; x \in K_n \text{ for all } n \in \mathbb{N} \right\}.$$

Show that A is compact.

Problem 5. **(6 pts)**
Let

$$S = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; x > 0 \text{ and } y < 0 \right\} \subset \mathbb{R}^2,$$

and let $f\colon S \to \mathbb{R}$ and $g\colon S \to \mathbb{R}$ be two continuous functions. Assume that there exists points $z, w \in S$ such that $f(z) < g(z)$ and $f(w) > g(w)$. Show that there exists a point $u \in S$ such that $f(u) = g(u)$.

Problem 6. **(6 pts)**

(a) For $n \in \mathbb{N}$, let $f_n\colon [0, \infty) \to \mathbb{R}$ be the function given by

$$f_n(x) = \frac{n}{1 + nx + n^2\,(x^2 - x)^2} \qquad \text{for } x \in [0, \infty).$$

Does the function sequence $(f_n)_{n\in\mathbb{N}}$ converge pointwise? Does the function sequence $(f_n)_{n\in\mathbb{N}}$ converge uniformly?

(b) For $n \in \mathbb{N}$, let $g_n\colon [1, \infty) \to \mathbb{R}$ be the function given by

$$g_n(x) = \frac{n}{1 + nx + n^2\,(x^2 - x)^2} \qquad \text{for } x \in [1, \infty).$$

Does the function sequence $(g_n)_{n\in\mathbb{N}}$ converge pointwise? Does the function sequence $(g_n)_{n\in\mathbb{N}}$ converge uniformly?

(c) For $n \in \mathbb{N}$, let $h_n\colon [2, \infty) \to \mathbb{R}$ be the function given by

$$h_n(x) = \frac{n}{1 + nx + n^2\,(x^2 - x)^2} \qquad \text{for } x \in [2, \infty).$$

Does the function sequence $(h_n)_{n\in\mathbb{N}}$ converge pointwise? Does the function sequence $(h_n)_{n\in\mathbb{N}}$ converge uniformly?