Abstract Algebra Exam, MS-C1081

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You may bring to the exam a memory aid sheet of size A4. The memory aid sheet must be hand-written, contain text on one side only, and have your name and student number written in the top right corner. You do not need to return the memory aid sheet.

- If you are taking MS-C1081 course exam (KT), then do problems 1-5. Your final points for the course will be exam points + exercise points. You may choose to do the option below, in which case your final points for the course will be maximum of the two options.
- If you are taking MS-C1081 general exam (T0), then do problems 1-6. Your final points for the course will be 5/3*(exam points).

Problems:

1. (a) (5 points) Let p be a fixed prime. Let R^p be the set of rationals whose denominator is a power of p (p^i, $i \geq 0$). Prove that R^p is an abelian group under ordinary addition of rationals.

   (b) (4 points) Let G be a group. The center of G is

   $$C = \{a \in G : ax = xa \text{ for all } x \in G\}.$$

   Prove that C is a subgroup of G.

2. Recall that S_3 is the set of all permutations of $\{1, 2, 3\}$ and it forms a group under taking compositions.

   (a) (5 points) Let H be the cyclic subgroup (of order 2) of S_3 generated by $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$. Show that no left coset of H (except H itself) is also a right coset.

   (b) (2 points) Let K be the cyclic subgroup (of order 3) of S_3 generated by $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$. Show that every left coset of K is also a right coset of K.

   (c) (4 points) Point (b) proves that K is a normal subgroup of S_3. Write down the Cayley table for the quotient group S_3/K.

3. (a) (5 points) Let $(G, \cdot)$ be any group and let a be any element of G. Let $\phi : \mathbb{Z} \to G$ be defined by $\phi(n) = a^n$. Show that ϕ is a homomorphism. Describe the image and the possibilities for the kernel of ϕ.
   (b) (5 points) Mark each of the following true or false. Justify your answer.
      i. For any two groups G and G', there exists a homomorphism of G into G'.
      ii. A group homomorphism may have an empty kernel.
      iii. It is not possible to have a nontrivial homomorphism of some finite group into some infinite group.
      iv. There is a nontrivial group homomorphism $\phi : \mathbb{Z}_{12} \to \mathbb{Z}_5$.
      v. There is a nontrivial group homomorphism $\phi : \mathbb{Z}_3 \to \mathbb{Z}$.

4. (a) (4 points) List all subgroups of the group $(\mathbb{Z}_{10}, +)$. Justify why the list of subgroups is complete.
   (b) (2 points) Determine the order of each element of $(\mathbb{Z}_{10}, +)$ and list the generators of $(\mathbb{Z}_{10}, +)$.
   (c) (4 points) Let G be a group and $\phi : \mathbb{Z}_{10} \to G$ a group homomorphism. What are the possible images $\phi(\mathbb{Z}_{10})$ of $\mathbb{Z}_{10}$ under the group homomorphism ϕ (up to isomorphism)? Justify your answer.

5. (a) (2 points) Decide whether the following operations of addition and multiplication are closed on the set, and give a ring structure. If a ring is formed, show that it is a ring and state whether the ring is commutative. If a ring is not formed, tell why this is the case.
      i. $\mathbb{Z}^+$ (the set of strictly positive integers) with the usual addition and multiplication.
      ii. The set of qi for $q \in \mathbb{Q}$ with the usual addition and multiplication. Here i denotes the imaginary unit.
   (b) (5 points) Let R, R' be rings and $f : R \to R'$ be a ring homomorphism. Let I be an ideal of R. Prove that $f(I)$ is an ideal of $f(R)$.
   (c) (3 points) For each of the following rings R, determine the zero-divisors and the set of units.
      i. The polynomial ring $\mathbb{R}[x]$.
      ii. $\mathbb{Z} \times \mathbb{Z}$, where addition and multiplication are defined componentwise.
      iii. $\mathbb{R} \times \mathbb{R}$, where addition and multiplication are defined componentwise.

6. (a) (6 points) Let R be a ring with more than one element such that for each nonzero $a \in R$ there is a unique $b \in R$ such that $aba = a$. Prove:
      i. R has no zero divisors.
      ii. $bab = b$.
      iii. R is a division ring.
   (b) (4 points) Prove that the fields $\mathbb{R}$ and $\mathbb{C}$ are not isomorphic.