Aalto University
School of Science

PHYS-E0420 Many-Body Quantum Mechanics
Midterm Exam 23.02.2022

1. Explain and describe briefly the following concepts. No derivations are needed, but you can use equations and mathematical formula when you find it useful. Pictures are also welcome.
   a) Oscillator strength and sum rules.
   b) Dipole approximation.
   c) Fock states. How would you draw a Fock state in a phase space picture?

2. Derive the Fermi golden rule for the (absorption) transition rate from the initial state $|\psi(t=0)\rangle = |i\rangle$ to a continuous part of the spectrum.

   a) Start from the first-order time-dependent perturbation theory result

$$\langle f|\psi(t)\rangle \approx \frac{1}{\hbar}\int_0^t dt' \, \langle f|H'(t')|i\rangle e^{i\omega_{fi}t'},$$

   where $|i\rangle$ and $|f\rangle$ are the initial and final states of the system, $\omega_{fi} = (E_f - E_i)/\hbar$, and $H'(t) = Ae^{-i\omega t} + A^\dagger e^{i\omega t}$ is a harmonic perturbation. Derive the result for resonant absorption probability

$$P_{if}(t) = \frac{1}{\hbar^2}|\langle f|A|i\rangle|^2 \left[\frac{\sin\left[\frac{1}{2}(\omega - \omega_{fi})\,t\right]}{\frac{1}{2}(\omega - \omega_{fi})}\right]^2, \quad \text{for} \quad \omega \approx \omega_{fi},$$

   You can assume that the perturbation is long lasting, such that $|\omega t| \gg 1$.

   b) Using the above transition probability, write the total transition rate out from the initial state $|i\rangle$ when there are several possible final states $|f_n\rangle$ with energies $E_{f,n} = \hbar\omega_n$. You can assume that the coupling $\langle f|A|i\rangle$ is equal for all final states $|f\rangle = |f_n\rangle$. Now write the total transition rate when the final states form a continuum with constant density of states $g(E) = g$.

   c) Solve the total transition rate and derive the Fermi golden rule. Since off-resonant (non-energy-conserving) processes are strongly suppressed, you are able to do some approximations. The following relation may also prove useful

$$\int_{-\infty}^{\infty}\left[\frac{\sin x}{x}\right]^2 dx = \pi.$$

3. The Hamiltonian in the field operator formulation is

$$\begin{aligned} H &= \int d^3x \left(\frac{\hbar^2}{2m}\nabla\psi^\dagger(\mathbf{x})\nabla\psi(\mathbf{x}) + U(\mathbf{x})\psi^\dagger(\mathbf{x})\psi(\mathbf{x})\right) \\ &+ \frac{1}{2}\int d^3x \int d^3x' \, \psi^\dagger(\mathbf{x})\psi^\dagger(\mathbf{x}')V(\mathbf{x},\mathbf{x}')\psi(\mathbf{x}')\psi(\mathbf{x}), \end{aligned}$$

where U is an external potential, m the mass of the particle, and V the interaction potential. Starting from the Heisenberg equation of motion

$$\hbar\frac{\partial}{\partial t}\psi(\mathbf{x},t) = -[H,\psi(\mathbf{x},t)] = -e^{iHt/\hbar}[H,\psi(\mathbf{x},0)]e^{-iHt/\hbar}$$

derive the equation of motion for the field operator:

$$\hbar\frac{\partial}{\partial t}\psi(\mathbf{x},t) = \left(-\frac{\hbar^2\nabla^2}{2m} + U(\mathbf{x})\right)\psi(\mathbf{x},t) + \int d^3x'\psi^\dagger(\mathbf{x}',t)V(\mathbf{x},\mathbf{x}')\psi(\mathbf{x}',t)\psi(\mathbf{x},t). \quad (1)$$

How does the form of this equation relate to the name "second quantization"?

4. Let us define the so called quadratures of a quantized electromagnetic field for a single frequency ω

$$\hat{q} = \sqrt{\frac{\hbar}{2\omega}}(\hat{a}+\hat{a}^\dagger), \qquad \hat{p} = -i\sqrt{\frac{\hbar\omega}{2}}(\hat{a}-\hat{a}^\dagger).$$

The quantum Hamiltonian of the field is

$$\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a},$$

and the classical Hamiltonian is

$$H_{cl} = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2.$$

The coherent state $|\alpha\rangle$ is defined by demanding that the expectation value of the quantum Hamiltonian in this state equals the classical Hamiltonian, when p and q in the classical Hamiltonian are replaced by the expectation values of $\hat{p}$ and $\hat{q}$ in the state $|\alpha\rangle$. Using this requirement show how the coherent states $|\alpha\rangle$ must behave when acted on by the creation and annihilation operators. Then, derive the expression of the coherent state in terms of Fock states (you may directly use that $\langle 0|\alpha\rangle = \exp(-|\alpha|^2)$, just state from which condition this is obtained from).

Give some well-known example of a coherent state. Explain (by words and/or pictures, equations not needed) what is the difference between a coherent and a squeezed state.

Tip: When deriving the Fock state expansion, remember the algebra of creation and annihilation operators and how to "operate" to the left (to the bra-vector) using the definition of the adjoint operator.