Aalto University
Department of Mathematics and Systems Analysis
MS-C1541 — Metric spaces, 2022/III

Exam T0 / KT
24.2.2022
K Kytölä & M Orlich

This is the exam sheet for both the final exam (T0) and the course exam (KT) of MS-C1541 Metric spaces. The grading is based on either

- 100% final exam (T0);
- 50% course exam (KT) + 50% exercises (during the period III course).

You can attempt both options, and the one leading to the more favorable grade is taken into account.

Depending on the option above, you should solve the following problems:

- **Final exam (T0):** Solve all five problems.
- **Course exam (KT):** Choose any four of the five problems.

(If you solve all problems, the best four are taken into consideration for the course completion option based on course exam + exercises.)

PROBLEMS

Problem 1.
Prove that the formula

$$\mathsf{d}(x,y) = \frac{|x-y|}{1+|x-y|} \qquad \text{for } x,y \in \mathbb{R}$$

defines a metric in $\mathbb{R}$. **(6 pts)**

Problem 2.
Let (X, d) be a metric space.

(a) Define what it means for a sequence $(x_n)_{n\in\mathbb{N}}$ in X to have $x \in X$ as its limit. **(2 pts)**
(b) Show directly from the definition that if both $x \in X$ and $x' \in X$ are limits of the same sequence $(x_n)_{n\in\mathbb{N}}$, then $x = x'$. **(4 pts)**

Problem 3.
Possible or not? In each subproblem below, give an example of the specified kind, or explain why such an example cannot exist. In your justifications for the requested properties or their impossibility you can (and should) use results from the course.

(a) A sequence $(f_n)_{n\in\mathbb{N}}$ of functions $f_n : X \to Y$ between two metric spaces X and Y, such that $(f_n)_{n\in\mathbb{N}}$ converges uniformly but does not converge pointwise. **(3 pts)**
(b) A continuous function $f : X \to Y$ between metric spaces X and Y, and an open subset $U \subset X$ whose image $f[U] \subset Y$ is not open. **(3 pts)**

Problem 4.

(a) Give an example of a set X and a function $g\colon X \to \mathbb{R}$ for which there does not exist a point $\tilde{p} \in X$ such that $g(p) \leq g(\tilde{p})$ for all $p \in X$. **(1 pt)**

(b) Consider the set

$$A := \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} |x| + |z| \leq 2 \qquad \text{and} \\ -\cos\left(\dfrac{z}{3+x^4}\right) \leq y \leq \cos\left(\dfrac{xyz}{1+2z^2}\right) \end{array} \right\}$$

and the function $f\colon A \to \mathbb{R}$ defined by

$$f\big((x,y,z)\big) = \frac{7x^3 + 13\cos(yz-1)}{4 + \sin(2+y) + \cos(x^5y^4z^3)} \qquad \text{for } (x,y,z) \in A \subset \mathbb{R}^3.$$

Show that there exists a point $(\tilde{x}, \tilde{y}, \tilde{z}) \in A$ such that $f\big((x,y,z)\big) \leq f\big((\tilde{x},\tilde{y},\tilde{z})\big)$ for all $(x,y,z) \in A$. **(5 pts)**

Hint: *You probably want to use some general arguments instead of direct calculations (the formulas involved might get slightly complicated in a calculation). It can be considered known that the functions* $\sin\colon \mathbb{R} \to \mathbb{R}$ *and* $\cos\colon \mathbb{R} \to \mathbb{R}$ *and multivariate polynomials and rational functions are continuous. Also any other results from the course can be used.*

Problem 5.

Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the function defined by

$$f\big((x,\, y)\big) = \left(e^{-y^2/10},\; 1 + \frac{x+y}{8}\right) \qquad \text{for } (x,y) \in \mathbb{R}^2.$$

(a) Show that for any $(x,y), (x',y') \in \mathbb{R}^2$ we have

$$\Big\| f\big((x,\,y)\big) - f\big((x',\,y')\big) \Big\| \;\leq\; \frac{3}{4}\, \big\|(x,y) - (x',y')\big\|,$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^2$. **(2 pts)**

(b) Define a sequence $(v_n)_{n\in\mathbb{N}}$ in the Euclidean plane $\mathbb{R}^2$ recursively by setting

$$v_0 = (x_0,\, y_0) := (0,\, 1) \qquad \text{and}$$

$$v_n = (x_n,\, y_n) := f\big((x_{n-1},\, y_{n-1})\big) \qquad \text{for } n \in \mathbb{N}.$$

Using results from the course, show that the sequence $(v_n)_{n\in\mathbb{N}}$ converges to a limit $\tilde{v} = (\tilde{x}, \tilde{y}) \in \mathbb{R}^2$, and that the coordinates of this limit satisfy the equations $\tilde{x} = e^{-\tilde{y}^2/10}$ and $\tilde{y} = \frac{8+\tilde{x}}{7}$. **(4 pts)**