PHYS-C0256 - Thermodynamics and Statistical Physics
Exam on Feb. 24, 2021

4 problems - 30 points

1. A gas obeys the following equation of state,

$$p(V - Nb) = Nk_BT \exp{(-\frac{a}{V - Nb})}, \tag{1}$$

where a and b are the parameters of the system and N is kept fixed.

(a) Find the Maxwell relation involving $(\frac{\partial S}{\partial V})_T$. (2 points)

(b) Use a differential for S to calculate $dU(T, V)$ for the internal energy U. Show that for the given equation of state U is a function of T only. (4 points)

2. **(a)** Consider the cycle shown in Fig. 1. Calculate work and heat in each leg expressed in the pressures and volumes at points A-D. (4 points)

(b) Consider ideal gas with N particles initially in volume V_i. Let the gas expand to final volume $3V_i$. Assume the gas is in perfect contact with a surrounding heat bath at constant temperature T. First find the heat absorbed from the bath based on the number of available microstates in the beginning and at the end. Then derive the same result based on ideal gas properties. (4 points)

3. **(a)** Show that for normal metal-insulator-superconductor NIS junction at low temperature T, $eV, k_BT \ll \Delta$, the following equation holds approximately for its current I versus voltage V dependence,

$$I = I_0 e^{-(\Delta - eV)/k_BT}, \tag{2}$$

with $I_0 = \frac{1}{eR_T}\sqrt{\frac{\pi \Delta k_B T}{2}}$, and Δ the superconducting gap. Show further that this equation gives temperature in the sense that

$$\frac{d}{dV}(\ln I) = \frac{e}{k_BT}, \tag{3}$$

where the slope is precisely $\frac{e}{k_BT}$, dictated just by constants of nature and temperature. (4 points)

(b) The cooling power of N in the same configuration is maximized at $V \simeq \Delta/e$. A more precise calculation shows that the current at this point is $I = 0.48\frac{\Delta}{eR_T}\sqrt{k_BT/\Delta}$ and the cooling power has value $\dot{Q}_{\text{NIS}} = 0.59\frac{\Delta^2}{e^2R_T}(k_BT/\Delta)^{3/2}$. Write the efficiency η of the NIS refrigerator at this optimum point. (3 points)

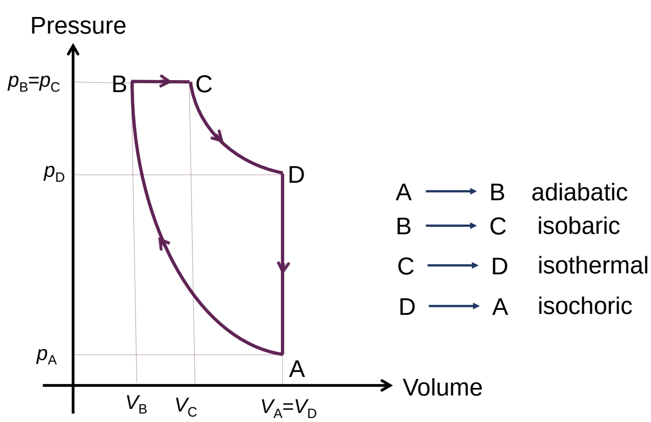

FIG. 1. Cycle of Problem 2(a).

4. Consider a two-level system (TLS) with energies $E_g = -E/2$ and $E_e = +E/2$ of the ground (g) and excited (e) states.

(a) Write the partition function Z for it, and populations of the states when the TLS is in equilibrium with a heat bath at temperature T. (2 points)

(b) Assume that the TLS is initiated in the ground state (non-equilibrium state), that is $p_g(0) = 1$ at $t = 0$. Let the transition rate from the ground state to the excited state be $\Gamma_\uparrow$ and that from the excited state to the ground state $\Gamma_\downarrow$. Find the time evolution of $p_g(t)$. Based on what you obtain for $p_g(\infty)$ and the result of (a), what can you say about the relation between $\Gamma_\uparrow$ and $\Gamma_\downarrow$ if they are determined by the coupling to the heat bath of (a)? (4 points)

(c) Assume the TLS represents a qubit. Let us drive it initially to a superposition state $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|g\rangle + |e\rangle)$. Find the time evolution of the density matrix components $\rho_{gg}(t)$ and $\rho_{ge}(t)$, with the initial condition $\rho(0) = |\psi(0)\rangle\langle\psi(0)|$ assuming they obey the standard master equation due to coupling to the bath at temperature T. (3 points)