Aalto University
School of Science

PHYS-E0420 Many-Body Quantum Mechanics
Second Exam 11.04.2022

1. Write a mini-essay (0.5-1 page each) about the following two topics:
   a) Rabi oscillations, Rabi splitting and avoided crossing. No derivations are needed, just use words, pictures and simple equations.

   b) Berry curvature and quantum metric (state the basic definitions by equations, no other derivations or calculations needed; in addition to the definitions, just use words and pictures for explaining the concepts)

2. Consider non-interacting bosons in a cube of volume $V = L^3$, in which L is the length of the cube. Show that there is Bose-Einstein condensation for certain particle numbers and temperatures.

   Guidelines: The bosons are described by the Hamiltonian

$$H = \int d\mathbf{x}\, \psi^\dagger(\mathbf{x}) \left(-\frac{\hbar^2\nabla^2}{2m}\right) \psi(\mathbf{x}). \tag{1}$$

   Assuming a periodic boundary condition, the single particle eigenstates are

$$\varphi_\mathbf{n}(\mathbf{x}) = \frac{1}{\sqrt{V}} e^{i\mathbf{k_n}\cdot\mathbf{x}} \tag{2}$$

   with the wave-vector $\mathbf{k_n} = \frac{2\pi}{L}(n_x, n_y, n_z)$, $n_l = 0, \pm 1, \pm 2, \ldots$ At an inverse temperature $\beta = \frac{1}{k_B T}$ and chemical potential μ the expected number of particles in state $\varphi_\mathbf{n}$ is

$$n_\mathbf{n} = \langle \hat{a}_\mathbf{n}^\dagger \hat{a}_\mathbf{n} \rangle = Tr\{\hat{\rho}\hat{a}_\mathbf{n}^\dagger \hat{a}_\mathbf{n}\} = \frac{ze^{-\beta E_\mathbf{n}}}{1 - ze^{-\beta E_\mathbf{n}}}, \tag{3}$$

   with the notation $z = e^{\beta\mu}$. The total density of particles in the system is

$$\frac{N}{V} = \frac{1}{V}\sum_\mathbf{n} n_\mathbf{n} = \frac{1}{V}\sum_\mathbf{n} \frac{ze^{-\beta E_\mathbf{n}}}{1 - ze^{-\beta E_\mathbf{n}}}. \tag{4}$$

   You may find it useful to estimate a discrete sum with an integral, to employ spherical coordinates and to use the following function

$$g_{\frac{3}{2}}(z) = \frac{4}{\sqrt{\pi}} \int_0^\infty dx\, x^2 \frac{ze^{-x^2}}{1 - ze^{-x^2}}, \qquad g_{\frac{3}{2}}(1) = 2.612\ldots \tag{5}$$

3. To describe e.g. bosonic atoms in an optical lattice one may use the Bose-Hubbard Hamiltonian

$$H = -J \sum_{<i,j>} a_i^\dagger a_j + \frac{U}{2} \sum_i a_i^\dagger a_i^\dagger a_i a_i$$

   together with the Gutzwiller mean-field ansatz

$$|\Psi_{MF}\rangle = \prod_{l=1}^{M} \left[\sum_{n_l=0}^{\infty} f_{n_l}^{(l)} |n_l\rangle \right].$$

a) Calculate the following quantities:

$$\langle \Psi_{MF} | H | \Psi_{MF} \rangle,$$
$$\langle \Psi_{MF} | \hat{n}_i | \Psi_{MF} \rangle,$$
$$\langle \Psi_{MF} | a_i | \Psi_{MF} \rangle,$$
$$\sigma_i^2 = \frac{\langle \Psi_{MF} | \hat{n}_i^2 | \Psi_{MF} \rangle - \langle \Psi_{MF} | \hat{n}_i | \Psi_{MF} \rangle^2}{\langle \Psi_{MF} | \hat{n}_i | \Psi_{MF} \rangle}.$$

(We have the notation $\hat{n}_i = a_i^\dagger a_i$.)

b) Explain briefly how you can distinguish between a Mott insulator and a superfluid phase based on the quantities above.

4. Consider the mean-field BCS Hamiltonian

$$H = \sum_{\mathbf{k}} \begin{pmatrix} c_{\mathbf{k}\uparrow}^\dagger & c_{-\mathbf{k}\downarrow} \end{pmatrix} \begin{pmatrix} \xi_{\mathbf{k}} & \Delta \\ \Delta & -\xi_{\mathbf{k}} \end{pmatrix} \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c_{-\mathbf{k}\downarrow}^\dagger \end{pmatrix}. \quad (6)$$

Diagonalize this by using the Bogoliubov transformation. Calculate the eigenenergies. You do not need to calculate the eigenvectors, we give them here:

$$u_{\mathbf{k}} = u_{\mathbf{k}\uparrow} = v_{\mathbf{k}\downarrow} = \sqrt{\frac{1}{2}\left(1 + \frac{\xi_{\mathbf{k}}}{\sqrt{\xi_{\mathbf{k}}^2 + \Delta^2}}\right)}, \quad (7)$$

$$v_{\mathbf{k}} = v_{\mathbf{k}\uparrow} = -u_{\mathbf{k}\downarrow} = \sqrt{\frac{1}{2}\left(1 - \frac{\xi_{\mathbf{k}}}{\sqrt{\xi_{\mathbf{k}}^2 + \Delta^2}}\right)}. \quad (8)$$

How are the new quasiparticle operators $\gamma_{\mathbf{k}\uparrow}$, $\gamma_{\mathbf{k}\downarrow}$ related to the original operators $c_{\mathbf{k}\uparrow}$, $c_{\mathbf{k}\downarrow}$? Calculate the momentum distribution $\langle n_{\mathbf{k}\uparrow} \rangle = \langle c_{\mathbf{k}\uparrow}^\dagger c_{\mathbf{k}\uparrow} \rangle$ and the order parameter $\Delta = -\frac{V_0}{V} \sum_{\mathbf{k}} \langle c_{\mathbf{k}\uparrow} c_{-\mathbf{k}\downarrow} \rangle$ using the BCS ansatz

$$|BCS\rangle = \prod_{\mathbf{k}} (u_{\mathbf{k}} + v_{\mathbf{k}} c_{-\mathbf{k}\downarrow}^\dagger c_{\mathbf{k}\uparrow}^\dagger) |0\rangle.$$

What is the functional dependence of $\langle n_{\mathbf{k}\uparrow} \rangle$ on $\mathbf{k}$ (e.g. sketch a figure) in the BCS and in the normal (non-interacting) state, and how does this relate to the concept of the Fermi surface? Note that $u_{\mathbf{k}}$, $v_{\mathbf{k}}$ in the BCS ansatz are the same as in the Bogoliubov tranformation.