Aalto University
Department of Mathematics and Systems Analysis
MS-E1200 - Lie groups and Lie algebras

Exam T0 / KT
Wed 13.4. at 13–16
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Allowed equipment:

- Writing equipment.
- A handwritten **memory aid sheet**. The memory aid sheet must be of size A4 with text only on one side, and it must contain your name and student number in the upper right corner.

The exam consists of 4 problems, each worth 6 points. Up to 6 bonus points from exercises are added to the exam score.

Problem 1. Consider the following set

$$G := \left\{ \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}.$$

(a) Show that G is a subgroup of the group $\mathrm{GL}_3(\mathbb{R})$ of invertible 3×3 real matrices. **(3 pts)**
(b) Show that $G \subset \mathrm{GL}_3(\mathbb{R})$ is a closed subset. **(1 pts)**
(c) Is G compact? **(1 pts)**
(d) Is G path-connected? **(1 pts)**
Recall: Path-connected means: for all $M, M' \in G$, there exists a continuous $\gamma \colon [0,1] \to G$ such that $\gamma(0) = M$ and $\gamma(1) = M'$.

Remark: Parts (a) and (b) show in particular that G is a matrix Lie group.

Problem 2. Let $n \in \mathbb{Z}_{>0}$. Consider the unitary group

$$\mathrm{U}_n = \left\{ M \in \mathbb{C}^{n \times n} \;\middle|\; M^\dagger M = \mathbb{I} \right\}.$$

(a) Denoting by $\exp \colon \mathbb{C}^{n \times n} \to \mathrm{GL}_n(\mathbb{C})$ the matrix exponential, define

$$\mathfrak{u}_n = \left\{ X \in \mathbb{C}^{n \times n} \;\middle|\; \forall t \in \mathbb{R} : \; \exp(tX) \in \mathrm{U}_n \right\}.$$

Show that $\mathfrak{u}_n$ consists of those $n \times n$ complex matrices X which satisfy $X^\dagger = -X$. **(2 pts)**
(b) Using the characterization of $\mathfrak{u}_n$ in part (a), show that $\mathfrak{u}_n$ is a vector subspace of the $\mathbb{R}$-vector space $\mathbb{C}^{n \times n}$ (the space of complex matrices is indeed viewed as a real vector space here). Calculate the dimension $\dim_{\mathbb{R}}(\mathfrak{u}_n)$. **(3 pts)**
(c) The centre of $\mathfrak{u}_n$ is by definition

$$\mathfrak{z} = \left\{ Z \in \mathfrak{u}_n \;\middle|\; \forall X \in \mathfrak{u}_n : \; [Z, X] = 0 \right\}.$$

Show that $\mathfrak{z} \neq \{0\}$. **(1 pts)**

Recall: $A^\dagger$ denotes the conjugate transpose of $A \in \mathbb{C}^{n \times n}$, with entries $(A^\dagger)_{ij} = \overline{A_{ji}}$.

Remark: Part (c) shows in particular that $\mathfrak{u}_n$ is not semisimple.

Problem 3. The symmetry group of the pentagon, i.e., the dihedral group D_5, is the group generated by elements r and m subject to relations $r^5 = e$, $m^2 = e$, and $mrmr = e$. The order of D_5 is 10. The conjugacy classes of D_5 are $\{e\}$, $\{r, r^4\}$, $\{r^2, r^3\}$, $\{m, mr, mr^2, mr^3, mr^4\}$.

(a) The following two invertible complex 2×2 matrices

$$R = \begin{bmatrix} \cos(\frac{2\pi}{5}) & -\sin(\frac{2\pi}{5}) \\ \sin(\frac{2\pi}{5}) & \cos(\frac{2\pi}{5}) \end{bmatrix} \qquad \text{and} \qquad M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

satisfy the relations $R^5 = M^2 = MRMR = \mathbb{I}$, so a representation of D_5 on $V = \mathbb{C}^2$ can be defined by specifying $r \mapsto R$ and $m \mapsto M$ for the generators. Compute the character of V, and show that V is irreducible. **(2 pt)**
(Recall that $\cos(\frac{2\pi}{5}) = \frac{\sqrt{5}-1}{4}$.)

(b) Show that there are two (mutually non-isomorphic) one-dimensional representations of D_5. Denote by U the trivial representation and by U' the other one. Compute the characters of U and U'. **(1 pt)**

(c) Find the character and dimension of the last irreducible representation of D_5, not isomorphic to the representations U, U', V in (a) and (b). **(2 pt)**

(d) A five-dimensional representation of D_5 can be defined by

$$r \mapsto \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \qquad m \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Find the multiplicity of each irreducible representation in the decomposition of this representation into a direct sum of irreducible representations. **(1 pt)**

Problem 4. Suppose that a complex Lie algebra $\mathfrak{g}$ has a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, which is abelian (i.e., $[H_1, H_2] = 0$ for all $H_1, H_2 \in \mathfrak{h}$), and that the Lie algebra $\mathfrak{g}$ admits a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \qquad \text{(vector space direct sum)}$$

where $\Phi \subset \mathfrak{h}^* \setminus \{0\}$ is a subset of non-zero elements of the the dual $\mathfrak{h}^*$, and

$$\mathfrak{g}_\alpha := \left\{ X \in \mathfrak{g} \;\middle|\; [H, X] = \alpha(H)\, X \quad \text{for all } H \in \mathfrak{h} \right\}.$$

(a) Show that if we have $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_{-\alpha}$ for some $\alpha \in \mathfrak{h}^*$ such that $\alpha, -\alpha \in \Phi$, then their bracket must satisfy $[X, Y] \in \mathfrak{h}$. **(2 pts)**

(b) Assume that we have $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_{-\alpha}$ for some $\alpha \in \Phi$ and assume moreover that $\alpha([X, Y]) \neq 0$. Let $\mathfrak{sl}_2(\mathbb{C})$ be the complex Lie algebra of traceless complex 2×2-matrices. Show that there exists a unique $c \in \mathbb{C} \setminus \{0\}$ such that for suitable $a, b \in \mathbb{C} \setminus \{0\}$, the linear map $\phi \colon \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g}$ given by

$$\phi\left(\begin{bmatrix} z & x \\ y & -z \end{bmatrix} \right) = ax\, X + by\, Y + cz\, [X, Y] \qquad \text{(for } x, y, z \in \mathbb{C}\text{)}$$

is a Lie algebra homomorphism. **(2 pts)**

(c) Let X, Y, a, b, c be as in part (b). Let $\vartheta \colon \mathfrak{g} \to \mathrm{End}(V)$ be a representation of $\mathfrak{g}$ in a finite-dimensional $\mathbb{C}$-vector space V. Using results from representation theory of $\mathfrak{sl}_2(\mathbb{C})$, show that any eigenvalue λ of the linear map $\vartheta([X, Y]) \colon V \to V$ must be an integer multiple of $1/c$, i.e., we have $\lambda \in c^{-1}\mathbb{Z}$. (State clearly which known results you used.) **(2 pts)**