Computational Algebraic Geometry Exam

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1. Let $I = \langle x, y^2 \rangle$ and $J = \langle x^2, y \rangle$ be ideals in $\mathbb{Q}[x, y]$. The main goal of the steps (a)-(f) is to compute $I \cap J$. Do all steps besides step (e) by hand. Give short explanations to your answers. You are encouraged to check your work using computer algebra software.

   (a) (1 point) Write down a generating set of size four for $tI + (1-t)J \subset \mathbb{Q}[t, x, y]$.

   (b) (2 points) Compute the S-polynomial $S(tx, (1-t)y)$ using the lex order with $t > x > y$.

   (c) (2 points) Compute the remainder on division of the S-polynomial in (b) by the generating set in (a) where generators are listed in any order.

   (d) (1 point) Is the generating set in (a) a Groebner basis wrt the lex order with $t > x > y$?

   (e) (2 points) Compute a Groebner basis of $tI + (1-t)J \subset \mathbb{Q}[t, x, y]$ wrt the lex order with $t > x > y$.

   (f) (1 point) Compute $I \cap J$.

   (g) (1 point) Compute $I + J$.

   (h) (1 points) Compute IJ.

   (i) (3 points) Compute $I^3 J^4 = IIIJJJJ$.

2. (12 points) Find the least common multiple and the greatest common divisor of

   - $4x^3 - 12x^2 + 11x - 3$,
   - $2x^3 - 9x^2 + 13x - 6$, and
   - $4x^3 - 24x^2 + 47x - 30$.

   You are allowed to use computer algebra software besides the commands that compute lcm, gcd and factorizations of polynomials directly. You are not allowed to use factorizations of polynomials derived by hand either. Explain your work.

3. (10 points) Consider the system of equations

$$x^3 + y^3 - z^3 = 1,$$
$$x^2 + y^2 + z^2 = 1,$$
$$x + y + z = 1.$$

Determine all solutions in $\mathbb{C}^3$ using the Elimination and the Extensions Theorems. You are allowed to use computer algebra software besides the commands that solve a system of polynomial equations directly. Explain your work.

4. (a) (5 points) Prove that every prime ideal is radical.
   (b) (5 points) Characterize prime monomial ideals. Explain your answer.
   (c) (2 point) Bring an example of a monomial ideal that is radical but not prime.
   (d) (2 points) Characterize varieties of monomial ideals and prime monomial ideals.