Instructions: Answer as many questions as possible. Each subquestion (labelled with letters) is worth a maximum of 6 points.

It is only permitted to bring to the exam room basic writing material and a scientific calculator.

Please write the solution to each problem (labelled with numbers) on a separate sheet of paper.

1. Let

$$A = \begin{bmatrix} 3 & 3 \\ -2 & -2 \\ -1 & -1 \\ 2 & 2 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 4 \\ -6 \\ 0 \\ 6 \end{bmatrix}.$$

(a) Prove that there exists a vector $\mathbf{z} \in \mathbb{R}^4$ and a vector $\mathbf{y} \in \mathbb{R}^2$ such that $A = \mathbf{z}\mathbf{y}^T$. Compute $\mathbf{z}$ and $\mathbf{y}$ explicitly. Then, define

$$\mathbf{u} := \frac{\mathbf{z}}{\|\mathbf{z}\|_2}, \mathbf{v} := \frac{\mathbf{y}}{\|\mathbf{y}\|_2}, \sigma := \|\mathbf{z}\|_2 \|\mathbf{y}\|_2$$

and explain why $A = \sigma \mathbf{u}\mathbf{v}^T$ is a singular value decomposition of A.

(b) Compute the nullspace of the matrix A and explain how to use the result to determine the number of solutions of the least square problem

$$\min_{\mathbf{x} \in \mathbb{R}^2} \|A\mathbf{x} - \mathbf{b}\|_2.$$

(c) Using part (a) or otherwise, compute the Moore-Penrose pseudoinverse of the matrix A. Then, compute the minimum norm solution to the least square problem $\min_{\mathbf{x} \in \mathbb{R}^2} \|A\mathbf{x} - \mathbf{b}\|_2$.

2. Let

$$B = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -y & 0 \\ 1 & 0 & y \end{bmatrix}$$

where $y \in \mathbb{R}$ is a parameter.

(a) Determine for which values of y the linear system $B\mathbf{x} = \mathbf{b}$ admits a unique solution for all possible right hand sides $\mathbf{b} \in \mathbb{R}^3$. Justify your answer.

(b) Give the definition of $R(M)$, the range of a matrix M. Then, compute a basis for $R(B)$ in the special case $y = 1$ and in the special case $y = -1$.

3. (a) Let $C \in \mathbb{R}^{2 \times 2}$. It is known that

$$N(C) = R(C) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

where $N(C)$ is the null space of C and $R(C)$ is the range of C. Prove that $C \neq 0$ and $C^2 = 0$. *Hint: a matrix* $M \in \mathbb{R}^{2\times 2}$ *satisfies* $M = 0$ *if and only if* $M\mathbf{v} = \mathbf{0}$ *for all* $\mathbf{v} \in \mathbb{R}^2$.

(b) Find the solution $\mathbf{x}(t) = e^{tD}\mathbf{x}(0)$ to the following initial value problem:

$$\mathbf{x}'(t) = D\mathbf{x}(t), \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}.$$

Hint: Observe that $N(D) = R(D) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$, *and thus, by part (a),* $D \neq 0 = D^2$.

4. Recall that $Q \in \mathbb{C}^{n\times n}$ is said to be a unitary matrix if $Q^*Q = QQ^* = I$ where Q^* denotes the conjugate transpose of Q.

   (a) Prove that if $Q \in \mathbb{C}^{n\times n}$ is a unitary matrix and $\mathbf{v} \in \mathbb{C}^n$ then $\|Q\mathbf{v}\|_2 = \|\mathbf{v}\|_2$. Then, explain why this implies that in the matrix operator 2-norm it holds $\|Q\|_2 = 1$. *Hint: For any vector* $\mathbf{w} \in \mathbb{C}^n$ *it holds* $\|\mathbf{w}\|_2^2 = \mathbf{w}^*\mathbf{w}$.

   (b) Using the statement of part (a) or otherwise, prove that if $Q, U \in \mathbb{C}^{n\times n}$ are both unitary matrices and $M \in \mathbb{C}^{n\times n}$ is any matrix, then $\|M\|_2 = \|QMU\|_2$.

   (c) Give the definition of an eigenvalue and an eigenvector of a matrix M. Then, prove the following fact: if $Q \in \mathbb{C}^{n\times n}$ is a unitary matrix and $\lambda \in \mathbb{C}$ is an eigenvalue of Q, then λ has modulus 1, i.e., $|\lambda| = \sqrt{\lambda^*\lambda} = 1$. *Hint:* $Q\mathbf{v} = \lambda\mathbf{v}$ *if and only if* $\mathbf{v}^*Q^* = \lambda^*\mathbf{v}^*$.