ELEC-E8101 Digital and Optimal Control

Final Exam (15.12.2021)

- *All four (4) problems must be answered.*
- *Clearly show your derivations and justifications. If you have made any extra assumption to solve the problem, clearly state that assumption in your response.*
- *You can write the solution as you wish in one (or more)* ***"pdf"*** *file, which contains the solutions such that it can be seen how you have solved the problems. You can use handwriting, but the result must be well readable. Please keep your solutions just like you made and submitted them, in case something went wrong with the submission or need to be checked afterwards.*
- *The exam is "open book" so you can have the course material (lectures, exercises, homework, Databook, etc) at your disposal. But you must not take help from any other person while taking the exam.*
- *Electronic calculators and computer software can be used, but* ***plots (if any) should be drawn in hand and justified****.*
- *With your signature (on the answer sheet you will submit) you assert that you have followed the above regulations.*

Good luck!

(Turn over)

1. a) Consider a unit-feedback system with open-loop transfer function given below:

$$F(z) = \frac{Y(z)}{U(z)} = \frac{K(z+1)}{z^2 - 1.6z + 0.6}$$

i) Compute the inverse z-transform of this transfer function $F(z)$ using partial fraction expansion. Show your calculations. [2p]

ii) Evaluate the "closed-loop" system stability using the Jury stability criterion. Find the values of gain K for which the closed-loop is stable. [2p]

b) Consider a causal, discrete-time, linear time-invariant system with transfer function

$$G(z) = \frac{z^2}{(z - a_1)(z - a_2)}$$

where $a_1 = 0.8e^{j\pi/6}$ and $a_2 = 0.8e^{-j\pi/6}$. In the following questions, *approximately* means that you should not use calculator, but rather give approximations and explain the approximations.

i) Draw the pole-zero diagram of $G(z)$. Is the system stable? [2p]

ii) Find the angle (or phase) at which the magnitude of $G(z)$, $|G(e^{j\omega})|$ obtains its maximum, denoted by $\omega_{\max}$. [2p]

iii) Approximately determine the magnitude $|G(e^{j\omega})|$ and phase $\angle G(e^{j\omega})$ at $\omega_{\max}$. [2p]

2. In the target tracking problems, the exact dynamics of the moving target is usually unknown. A typical solution is to approximately model the sampled-data target dynamics (in discrete-time) assuming it has nearly constant velocity in the 2D space. The dynamics, then, becomes in the state-space form as,

$$z(k+1) = Fz(k) + \Gamma u(k), \; z(k) = \begin{pmatrix} p_x(k) \\ p_y(k) \\ v_x(k) \\ v_y(k) \end{pmatrix}, \; F = \begin{pmatrix} 1 & 0 & T & 0 \\ 0 & 1 & 0 & T \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; \Gamma = \begin{pmatrix} \frac{T^2}{2} & 0 \\ 0 & \frac{T^2}{2} \\ T & 0 \\ 0 & T \end{pmatrix}$$

where the vector z represents the positions p_x and p_y and velocities v_x and v_y of the target in the x and y directions in 2D plane, respectively, and constant T is the sampling period. Similarly, input u has two entries in u_x and u_y in x and y directions. Assume that the measured variables (the outputs) are the position of the target and,

$$y(k) = Cz(k) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} z(k)$$

In the following questions, justify your answers for different values of the sampling period T (if needed).

a) Find the difference equation for both position and velocity only for one dimension, for example, for the x-coordinates. Then, find the discrete-time (impulse) transfer function. Assume zero initial conditions. [3p]
Hint: notice that the dynamics for x and y coordinates are decoupled.

b) Comment on the stability of the system, is it stable, marginally stable, or unstable? justify your answer. [1p]

c) Is the system observable (or detectable) and reachable (or controllable)? Justify your answer. [3p]

d) For sampling $T = 2$, design an observer to locate the closed-loop poles at $p_1 = 0.5$ and $p_2 = 0.5$. [2p]

e) Design a Luenburger observer with deadbeat strategy. [2p]

f) For general sampling period T, design a deadbeat controller for stability over finite time-steps. How many steps are needed for the controller to drive the closed-loop system to zero? [3p]

(Turn over)

3. Consider a controller in the (continuous-time) form,

$$H(s) = \frac{42(s + 4.4)}{s + 18.5}$$

For the questions below clearly show your calculations or explain how you got the answer.

a) Find the frequency ω_0 at which the the phase of $H(j\omega)$ is maximum value. Find the frequency and gain of $H(j\omega)$ at ω_0. [3p]

b) Find the gain and phase as $\omega \to \infty$. [1p]
Hint: use the geometric position of the poles and zeros.

c) For sampling periods $T = 150msec$ ($msec$ is millisec), find the discretized transfer function $H(z)$ using the following approximate discretization methods. For each case, discuss the "stability" of the discretized system.

i) Forward-difference method [2p]

ii) Backward-difference method [2p]

iii) Tustin (or bilinear) method. [2p]

* **For Bonus Point:** For this case find the frequency-warping at ω_0 determined in part (a). Assume that a signal of the same frequency ω_0 is filtered by both $H(s)$ and its Tustin discretization $H(z)$. Do you think the digitally-filtered outcome would be distorted (any different) from the analog-filtered output signal. Explain your answer. [**Extra** 2p]

iv) ZOH equivalent approximation. [2p]

v) Pole-zero mapping. For this case, set the gain of the discretized system the same as the gain of $H(s)$ at zero frequency ($\omega = 0$). [2p]
Hint: for pole-zero mapping, you map all the poles and zeros of the analog transfer function using the unique mapping between the s-domain and the z-domain, and you use those poles and zeros in the digital transfer function and, then, design the gain accordingly.

For all the above steps you need to clearly show your calculations and derivations in your response.

4. A magnetic levitation system controls the position of a ball in mid-air using the magnetic field generated by an electric current I in a coil wrapped around an iron core, as in Fig. 1.

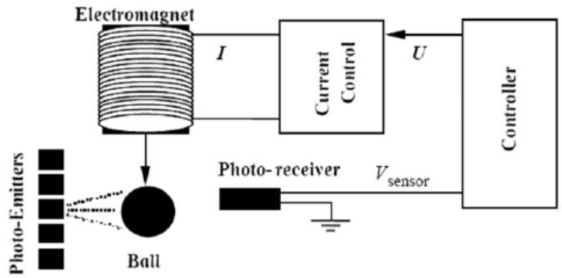

Figure 1: A Magnetic-levitation-system-diagram

Equation of motion for the ball is $m\ddot{x} = mg + f(x, I)$, where m is the mass, g is the gravitational constant, and $f(x, I)$ represents the electromagnetic force. Linearizing the equation about the equilibrium point x_0 (with electromagnetic force at a current balancing the gravitational force) we obtain:

$$m\ddot{x} = k_1 x + k_2 i$$

where x represents the displacement about the equilibrium point x_0 and $i = I - I_0$. For this question assume the following parameter values: $m = 0.01kg$, $k_1 = 10\frac{N}{m}$, and $k_2 = 0.2\frac{N}{A}$.

a) Compute the analog transfer function and comment on the open-loop stability. [2p]

$$G(s) = \frac{X(s)}{I(s)}$$

b) Assuming the sampling period equal to $T = 20\ msec$, compute the discretized transfer function $G(z)$ using impulse invariant method. [2p]

c) Consider the proportional controller with gain K_p in the forward path (open-loop). Approximately draw the Nyquist diagram based on the location of open-loop poles for $K_p = 1$. Discuss stability and number of unstable closed-loop poles using the Nyquist criteria for different values of K_p. Find the values of K_p for which the closed-loop system is stable (if any). Show your calculations and how you draw the Nyquist diagram. [4p]

d) A control engineer designed the following controller instead,

$$D(z) = 114.2\frac{z - 0.53}{z - 0.075}$$

to get the following specifications for the closed-loop step response: rise time $t_r < 0.1s$, settling time $t_s < 0.4s$, overshoot $M_p < 20\%$. For this designed controller, find the closed loop characteristic polynomial and briefly comment on the closed-loop stability. Check with some calculations if the specifications are met and justify your answer. [4p]

e) **For Bonus point:** What is special about this controller $D(z)$ and why do you think it is considered for this system transfer function G(z)? [**Extra** 1p]