MS-E1461 Hilbert spaces (Aalto University, Turunen / Vavilov)
Examination on Monday 17.10.2022, at 9:00-12:00

Points also for good effort! No literature, no calculators, etc. If you refer to any results in the course, please name them.

Note: In the following, H is an infinite-dimensional **complex** Hilbert space. with inner product $(u, v) \mapsto \langle u, v\rangle$ and norm $u \mapsto \|u\| = \langle u, u\rangle^{1/2}$.

1. Let $u, v \in \ell^2(\mathbb{Z}^+)$. Prove the following inequalities:

   (a) $|\langle u, v\rangle| \leq \|u\| \, \|v\|$.

   (b) $\|u + v\| \leq \|u\| + \|v\|$.

2. (a) Let $E \subset \mathbb{Z}$ and define $P : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ by $Pu(x) := u(x)$ for $x \in E$, and $Pu(x) := 0$ for $x \notin E$. Show that P is an orthogonal projection.

   (b) Find a countable dense subset of Hilbert space $\ell^2(\mathbb{Z})$.

   (c) Let $P, Q : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be orthogonal projections. Show that PQ is an orthogonal projection if and only if $PQ = QP$.

3. Let us consider a complex infinite-dimensional Hilbert space H.

   (a) Suppose $\langle u, v\rangle = \langle w, v\rangle$ for all $v \in H$. Show that $u = w$.

   (b) Show that $\varphi(u) := \langle u, v\rangle$ defines a linear functional $\varphi : H \to \mathbb{C}$. Moreover, show that $\|\varphi\| = \|v\|$.

   (c) Suppose $(u_k)_{k=1}^{\infty}$ is an orthonormal sequence in H. Show that $\lim_{k\to\infty} \langle u_k, v\rangle = 0$ for all $v \in H$.

4. Let $f : \mathbb{Z}^+ \to \mathbb{R}$ be a bounded injection. Let $P_k : H \to H$ be an orthogonal projection for each $k \in \mathbb{Z}^+$ such that $P_k \neq 0 = P_j P_k$ for each $j, k \in \mathbb{Z}^+$.

   (a) Show that $A : H \to H$ is linear and bounded, when

   $$Au := \sum_{k=1}^{\infty} f(k) \, P_k u.$$

   (b) Show that A is self-adjoint.

   (c) Under which condition A is positive? Why?

   (d) Under which condition A is unitary? Why?

   (e) Under which condition A is invertible? Why?

   (f) Under which condition A is compact? Why?