Mathematics
Aalto University
Hannukainen/Autio

MS-E1651 - Numerical Matrix Computations

Course exam/Exam 19.10.2022

Please fill in clearly *on every sheet* your name, student number, and the course code.

Solve all problems. The grade is the best of two: (i) exercise points and exam points (60/40) (ii) exam only. In both cases you must obtain at least grade 1 from the exam.

1. Let $A \in \mathbb{R}^{5\times 5}$ be s.p.d.. The non-zero entries of A are

$$\begin{bmatrix} \# & \# & 0 & 0 & 0 \\ \# & \# & \# & 0 & \# \\ 0 & \# & \# & \# & 0 \\ 0 & 0 & \# & \# & 0 \\ 0 & \# & 0 & 0 & \# \end{bmatrix}.$$

   (a) (2p) Draw the graph of A.

   (b) (2p) Give the non-zero structure of the Cholesky factor of A.

   (c) (2p) Explain what is fill-in and how it can be reduced.

2. Let $P \in \mathbb{R}^{3\times 3}$ be a permutation matrix s.t. $P\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^T = \begin{bmatrix} v_2 & v_1 & v_3 \end{bmatrix}^T$. In addition, let

$$P^T A P = LL^T \quad \text{where} \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}.$$

   (a) (3p) Write down the entries of P and solve the linear system $A\boldsymbol{x} = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}^T$.

   (b) (3p) Let $\boldsymbol{b}_1 \in \mathbb{R}^n$ and $\tilde{L} \in \mathbb{R}^{n\times n}$ be the Cholesky factor of $C \in \mathbb{R}^{n\times n}$. Compute the Cholesky factorization of

$$B = \begin{bmatrix} 1 & \boldsymbol{b}_1^T \\ \boldsymbol{b}_1 & C + \boldsymbol{b}_1\boldsymbol{b}_1^T \end{bmatrix}.$$

   Hint: proceed as in the existence proof.

3. (a) (2p) Give the definition of a positive definite matrix. Let $E \in \mathbb{R}^{n\times n}$ be s.t. $E = E^T$ and $\|E\|_2 = 2$. For which $\alpha \in \mathbb{R}$ can you guarantee that

$$\alpha I + E$$

   is symmetric and positive definite? Hint: use the properties of the operator norm and the inequality $-\boldsymbol{v}^T E\boldsymbol{w} \geq -\|\boldsymbol{v}\|_2 \|E\boldsymbol{w}\|_2$ for any $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$.

   (b) (3p) Let $A_1, A_2, A_3 \in \mathbb{R}^{10\times 10}$ be defined as

$$A_1 = QDQ^T, \quad A_2 = A_1^2, \quad \text{and} \quad A_3 = dI + A_1$$

   Here $Q \in \mathbb{R}^{10\times 10}$ is a unitary matrix, $D \in \mathbb{R}^{10\times 10}$ is a diagonal matrix with entries $D_{ii} = i$, and $d = 10^6$. Consider solving systems $A_i\boldsymbol{x} = \boldsymbol{b}$ for $i \in \{1, 2, 3\}$ in floating-point representation. Order the systems based on how accurately you expect that they can be solved. Justify your ordering.

(c) (1p) Name one direct and one iterative method for the solution of the s.p.d. linear system $A\boldsymbol{x} = \boldsymbol{b}$. Explain briefly how these methods work and why you chose them.

4. Let S be a subspace of $\mathbb{R}^n$ and

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\} \subset \mathbb{R}^n$$

a basis of S. In addition let $A \in \mathbb{R}^{n \times n}$ and $\boldsymbol{b} \in \mathbb{R}^n$ be defined as

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

(a) (2p) Compute an A-orthonormal basis $\{\boldsymbol{p}_0, \boldsymbol{p}_1\}$ for S.

(b) (4p) Let $A\boldsymbol{x} = \boldsymbol{b}$. Find the best possible approximation to $\boldsymbol{x}$ from S in the A-norm. You are *not* allowed to solve $\boldsymbol{x}$. Hint: The formula of A-orthogonal projection to space $R(Q)$ is $P_A = Q(Q^T AQ)^{-1}Q^T A$ (here Q is of full rank).

Some useful identities: Let $A \in \mathbb{R}^{n \times n}$, $A = A^T$, and denote by λ_m and λ_M the smallest and largest eigenvalue of A, respectively. Then

$$\lambda_m = \min_{v \in \mathbb{R}^n} \frac{\boldsymbol{v}^T A \boldsymbol{v}}{\boldsymbol{v}^T \boldsymbol{v}}, \quad \lambda_M = \max_{v \in \mathbb{R}^n} \frac{\boldsymbol{v}^T A \boldsymbol{v}}{\boldsymbol{v}^T \boldsymbol{v}}. \quad \|A\|_2 = \lambda_M, \quad \kappa_2(A) = \lambda_M \lambda_m^{-1}$$