ELEC-E4130 Electromagnetic fields **Midterm 1, October 20, 2022**

Answer all questions on the provided separate answer sheets. No calculators or own papers are allowed. Submit at least one answer sheet with your name and student number. All answer sheets must be returned, including sketches and empty ones. You can keep this question sheet.

1. Answer the following briefly (2 points each). Verbal explanations are preferred over formulas.
   (a) State the *divergence theorem* in words.
   (b) Explain *Lenz's law.*
   (c) What is meant by the *polarization* of a plane wave? When is a wave circularly polarized?
   (d) Define *critical angle.* When does it exist at an interface between two non-magnetic media?

2. All electromagnetic fields satisfy Maxwell's equations in any medium for any sources. Instead of solving these equations simultaneously, we can formulate separate wave equations for **E** and **H**, but several assumptions about the medium and sources are needed to get there. Derive the homogeneous vector wave equation

$$\nabla^2\mathbf{E} - \frac{1}{u^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = \mathbf{0}, \qquad u = \frac{1}{\sqrt{\varepsilon\mu}}$$

starting from Maxwell's equations

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \qquad \nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}, \qquad \nabla\cdot\mathbf{D} = \rho, \qquad \nabla\cdot\mathbf{B} = 0$$

and describe all assumptions that are needed at each step in the process. (8 points)

3. Medium 1 ($z < 0$) is free space and medium 2 ($z > 0$) is a lossless dielectric with $\varepsilon_r = 4$. The incident wave in medium 1 is given by the vector phasor

$$\mathbf{E}_i = \mathbf{a}_y E_0 e^{-j\beta_1 z}, \qquad \beta_1 = k_0 = \omega\sqrt{\mu_0\varepsilon_0}.$$

   (a) Determine the reflection and transmission coefficients. (3 points)
   (b) Express the total electric field (vector phasor) in medium 1. (3 points)
   (c) Determine the standing-wave ratio in medium 1. (2 points)

Vector identities from the back cover of Cheng's book:

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \mathbf{B}\cdot\mathbf{C}\times\mathbf{A} = \mathbf{C}\cdot\mathbf{A}\times\mathbf{B}$$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

$$\nabla(\psi V) = \psi\nabla V + V\nabla\psi$$

$$\nabla\cdot(\psi\mathbf{A}) = \psi\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla\psi$$

$$\nabla\times(\psi\mathbf{A}) = \psi\nabla\times\mathbf{A} + (\nabla\psi)\times\mathbf{A}$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

$$\nabla\cdot\nabla V = \nabla^2 V$$

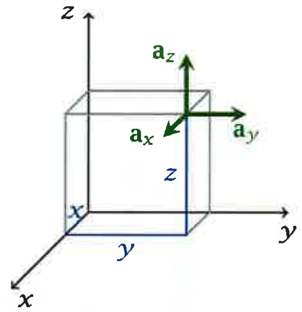
$$\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$$

$$\nabla\times(\nabla V) = \mathbf{0}$$

$$\nabla\cdot(\nabla\times\mathbf{A}) = 0$$

Nabla operations

Cartesian coordinates (x, y, z)



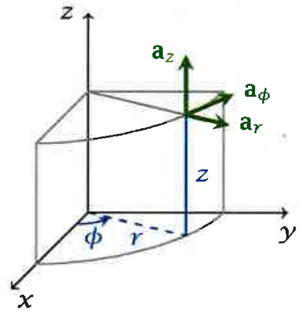
$$\nabla V = \mathbf{a}_x \frac{\partial V}{\partial x} + \mathbf{a}_y \frac{\partial V}{\partial y} + \mathbf{a}_z \frac{\partial V}{\partial z}$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Cylindrical coordinates (r, ϕ, z)



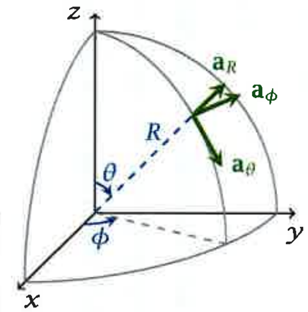
$$\nabla V = \mathbf{a}_r \frac{\partial V}{\partial r} + \mathbf{a}_\phi \frac{1}{r}\frac{\partial V}{\partial \phi} + \mathbf{a}_z \frac{\partial V}{\partial z}$$

$$\nabla \times \mathbf{A} = \frac{1}{r}\begin{vmatrix} \mathbf{a}_r & \mathbf{a}_\phi r & \mathbf{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

Spherical coordinates (R, θ, ϕ)



$$\nabla V = \mathbf{a}_R \frac{\partial V}{\partial R} + \mathbf{a}_\theta \frac{1}{R}\frac{\partial V}{\partial \theta} + \mathbf{a}_\phi \frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \frac{1}{R^2 \sin\theta}\begin{vmatrix} \mathbf{a}_R & \mathbf{a}_\theta R & \mathbf{a}_\phi R\sin\theta \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_R & RA_\theta & (R\sin\theta)\,A_\phi \end{vmatrix}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 A_R\right) + \frac{1}{R\sin\theta}\frac{\partial}{\partial \theta}\left(A_\theta \sin\theta\right) + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

Coordinate transformations

Cartesian ↔ Cylindrical

$$x = r\cos\phi, \qquad y = r\sin\phi, \qquad z = z$$

$$r = \sqrt{x^2+y^2}, \qquad \phi = \tan^{-1}\frac{y}{x}, \qquad z = z$$

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} A_r \\ A_\phi \\ A_z \end{pmatrix}$$

$$\begin{pmatrix} A_r \\ A_\phi \\ A_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

Cartesian ↔ Spherical

$$x = R\sin\theta\cos\phi, \qquad y = R\sin\theta\sin\phi, \qquad z = R\cos\theta$$

$$R = \sqrt{x^2+y^2+z^2}, \qquad \theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}, \qquad \phi = \tan^{-1}\frac{y}{x}$$

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}\begin{pmatrix} A_R \\ A_\theta \\ A_\phi \end{pmatrix}$$

$$\begin{pmatrix} A_R \\ A_\theta \\ A_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix}\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

Cylindrical ↔ Spherical

$$r = R\sin\theta, \qquad \phi = \phi, \qquad z = R\cos\theta$$

$$R = \sqrt{r^2+z^2}, \qquad \theta = \tan^{-1}\frac{r}{z}, \qquad \phi = \phi$$

$$\begin{pmatrix} A_r \\ A_\phi \\ A_z \end{pmatrix} = \begin{pmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}\begin{pmatrix} A_R \\ A_\theta \\ A_\phi \end{pmatrix}$$

$$\begin{pmatrix} A_R \\ A_\theta \\ A_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} A_r \\ A_\phi \\ A_z \end{pmatrix}$$

Other useful formulas

Cartesian coordinates

$$d\boldsymbol{\ell} = \mathbf{a}_x\,dx + \mathbf{a}_y\,dy + \mathbf{a}_z\,dz$$
$$ds_x = dy\,dz$$
$$ds_y = dx\,dz$$
$$ds_z = dx\,dy$$
$$dv = dx\,dy\,dz$$

Cylindrical coordinates

$$d\boldsymbol{\ell} = \mathbf{a}_r\,dr + \mathbf{a}_\phi r\,d\phi + \mathbf{a}_z\,dz$$
$$ds_r = r\,d\phi\,dz$$
$$ds_\phi = dr\,dz$$
$$ds_z = r\,dr\,d\phi$$
$$dv = r\,dr\,d\phi\,dz$$

Spherical coordinates

$$d\boldsymbol{\ell} = \mathbf{a}_R\,dR + \mathbf{a}_\theta R\,d\theta + \mathbf{a}_\phi R\sin\theta\,d\phi$$
$$ds_R = R^2\sin\theta\,d\theta\,d\phi$$
$$ds_\theta = R\sin\theta\,dR\,d\phi$$
$$ds_\phi = R\,dR\,d\theta$$
$$dv = R^2\sin\theta\,dR\,d\theta\,d\phi$$

Divergence theorem $\int_V \nabla\cdot\mathbf{A}\,dv = \oint_S \mathbf{A}\cdot d\mathbf{s}$

Stokes' theorem $\int_S (\nabla\times\mathbf{A})\cdot d\mathbf{s} = \oint_C \mathbf{A}\cdot d\boldsymbol{\ell}$

Constants

$$c = 299\,792\,458\,\frac{\mathrm{m}}{\mathrm{s}}$$

$$\mu_0 = 4\pi\times10^{-7}\,\frac{\mathrm{Vs}}{\mathrm{Am}} \approx 1.257\times10^{-6}\,\frac{\mathrm{H}}{\mathrm{m}}$$

$$\varepsilon_0 = \frac{1}{\mu_0 c^2} \approx 8.854\times10^{-12}\,\frac{\mathrm{As}}{\mathrm{Vm}} \quad \left(=\frac{\mathrm{F}}{\mathrm{m}}\right)$$

$$e \approx 1.602\times10^{-19}\,\mathrm{C}$$