ELEC-A7200 Signals and Systems

Autumn 2022, Midterm exam 1

October 21th 2022

1 Task

a) (2p.) Let $x_1(t)$ and $x_2(t)$ be orthonormal signals. Determine

$$\langle x_1(t) - x_2(t), x_1(t)\rangle \quad \text{and} \quad \langle x_1(t) - x_2(t), x_2(t)\rangle$$

b) (2p.) Let $x_3(t) = 2 \cdot \text{tria}\left(\frac{t}{4}\right)$, where

$$\text{tria}(t) = \begin{cases} 1+t, & -1 \le t < 0 \\ 1-t, & 0 \le t \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Express the signal $x_4(t) = \frac{dx_3(t)}{dt}$ using rectangular pulses of the form $\text{rect}\left(\frac{t-t_0}{T}\right)$, where

$$\text{rect}(t) = \begin{cases} 1, & |t| \le \frac{1}{2} \\ 0, & |t| > \frac{1}{2}. \end{cases}$$

Draw the signals $x_3(t)$ and $x_4(t)$.

c) (2p.) Calculate

$$\int_{-\infty}^{\infty} x_3(t)\,\delta(t-2)\,dt,$$

where $\delta(t)$ is the Dirac delta function.

d) (4p.) Let $x_5(t) = e^{-t}u(t)$, where $u(t)$ is the unit step function:

$$u(t) = \begin{cases} 1, & t \ge 0 \\ 0, & t < 0. \end{cases}$$

Calculate the convolution of the signal $x_5(t)$ with itself:

$$y(t) = \int_{-\infty}^{\infty} x_5(\tau)x_5(t-\tau)\,d\tau.$$

Hint: it is probably a good idea to draw a picture.

2 Task

Consider the periodic signal $x(t) = 2\cos(20\pi t)$.

a) (1p.) What are the period T_0 and the frequency of the signal $x(t)$?

b) (3p.) Determine the coefficients of the exponential Fourier series of this signal:

$$x_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-j2\pi \frac{k}{T_0} t} dt.$$

c) (2p.) Draw the two-sided amplitude spectrum and phase spectrum of this signal.

d) (2p.) Draw the one-sided power spectral density of this signal.

e) (2p.) Compute the average power of this signal.

3 Task

Consider the pulse $x(t) = \frac{A}{2}\text{rect}\left(\frac{t}{4}\right) + \frac{A}{2}\text{rect}\left(\frac{t}{2}\right)$. This pulse is shown in Figure 1.

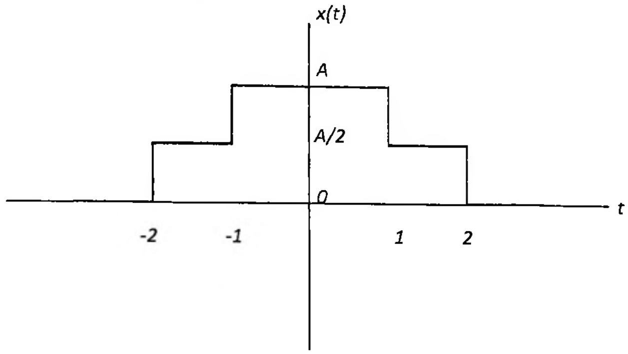

Figure 1: Pulssi/puls/pulse $x(t)$.

a) (2p.) Determine the amplitude A of the pulse so that the energy of the pulse is 1J.

b) (4p.) Find the Fourier transform $X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$ and the energy spectral density $|X(f)|^2$ of the signal $x(t)$.

c) The pulse $x(t)$ goes through an echoing channel. At the other end of the channel, the pulse $y(t)$ is received, which can be expressed as the convolution of the signal $x(t)$ and the so-called *impulse response* $h(t)$ of the channel:

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\, d\tau.$$

The impulse response is: $h(t) = \delta(t) + \delta(t-2)$, where $\delta(t)$ is the Dirac delta function.

i) (2p.) Find the Fourier transform $H(f)$ of the impulse response $h(t)$.

ii) (2p.) Find the Fourier transform $Y(f)$ and the energy spectral density $|Y(f)|^2$ of the signal $y(t)$.