ELEC-C9420 Introduction to Quantum Technologies, Fall 19
Midterm exam 2, 13.12.2019
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Instructions: Read carefully before you start working on the exam!

- Allowed tools: writing equipment, one A4-sized hand-written cheat sheet.
- The exam consists of three compulsory exam problems. Max 10 points per problem. Write your answers on the official answer sheets.
- Full points require explanations, not only computations!
- The answers must be given in terms of those quantities, for which symbols are given in the problem description.
- Remember to write your name on all the exam sheets you use. Prepare to prove your identity when you hand over your answers.

Problem 1

Explain each term or concept with 2-3 full sentences.

a) measurement postulate b) uncertainty principle c) Ehrenfest theorem
d) no-cloning theorem e) photon polarization

Problem 2

Consider a quantum particle on the real-line described by the wavefunction

$$\psi(x) = \begin{cases} Nx\sqrt{1-x^2} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

where $N = \sqrt{\frac{15}{2}}$ is the normalization constant.

a) What is the probability for finding the particle within the interval $0 < x < \frac{1}{2}$?
b) Compute the expectation value of the particle's position.
c) Compute the variance of the particle's position.
d) Compute the expectation value of the particle's momentum.

Problem 3

Consider a single qubit with the initial state $|0\rangle$ and the Hamiltonian operator $\hat{H}$, which acts on the basis states as $\hat{H}|0\rangle = |0\rangle + i|1\rangle$ and $\hat{H}|1\rangle = -i|0\rangle + |1\rangle$.

a) Show that the states

$$|E=0\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)\,, \qquad |E=2\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

are eigenstates of the Hamiltonian with the eigenvalues $E = 0$ and $E = 2$.
b) Find the state of the qubit at time t.
c) What is the probability of finding the qubit in state $|0\rangle$ at time t?
d) Let's say the qubit was observed to be in the state $|0\rangle$ at time t. What is the probability for the qubit to be in the state $|0\rangle$ at a later time $t + t'$ if it keeps evolving according to the given Hamiltonian?

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Midterm exam 2, example solutions
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Problem 1

a) Measurement postulate of quantum mechanics dictates that, when some observable of a quantum system associated with the operator $\hat{O}$ is measured, the quantum state of the system 'collapses' to an eigenstate of $\hat{O}$ associated with the eigenvalue, which was observed in the measurement. (1p) The measurement postulate indicates that, unlike for classical systems, the act of measurement significantly affects the time-evolution of a quantum system. (1p)

b) Uncertainty principle of quantum mechanics says that the values of 'complementary' observables, such as position and momentum, cannot be exactly known simultaneously, but the product of their variances is bounded from below. (1p) The uncertainty principle is due to the fact that there do not exist common eigenstates for non-commuting observables. (1p)

c) Ehrenfest theorem shows that, although quantum particles behave probabilistically, the expectation values of their position and momentum satisfy Newton's classical equations of motion. (1p) Therefore, even though we need quantum mechanics to describe microscopic particles, the behavior of macroscopic objects can be described very well by classical mechanics. (1p)

d) No-cloning theorem for quantum states shows that it is impossible to find a quantum algorithm or operation, which would perfectly copy an arbitrary quantum state from one quantum system to another without destroying the original copy. (2p)

e) Just like classical electromagnetic waves, individual photons have polarization, which corresponds to the direction of the oscillations in the electric field. (1p) Photons have two orthogonal polarization states, for example, vertical and horizontal, and any other polarization state can be obtained as a superposition of these two. (1p)

Problem 2

a) Probability to find the particle within the interval is obtained as

$$\begin{aligned}\int_0^{\frac{1}{2}} |\psi(x)|^2 \, \mathrm{d}x &= N^2 \int_0^{\frac{1}{2}} x^2(1-x^2) \, \mathrm{d}x \\ &= N^2 \int_0^{\frac{1}{2}} (x^2 - x^4) \, \mathrm{d}x \\ &= \frac{15}{2}\left[\frac{1}{3}\left(\frac{1}{2}\right)^3 - \frac{1}{5}\left(\frac{1}{2}\right)^5\right] = \frac{7}{32}\,. \qquad \text{(2p)}\end{aligned}$$

b) Expectation value of the position is obtained as

$$
\begin{aligned}
\langle\psi|\hat{x}|\psi\rangle &= \int_0^1 x|\psi(x)|^2 \,\mathrm{d}x \\
&= N^2 \int_0^1 x^3(1-x^2) \,\mathrm{d}x \\
&= N^2 \int_0^1 (x^3 - x^5) \,\mathrm{d}x \\
&= \frac{15}{2}\left(\frac{1}{4}-\frac{1}{6}\right) = \frac{5}{8}\,. \qquad \text{(2p)}
\end{aligned}
$$

c) Expectation value of the square of the position is obtained as

$$
\begin{aligned}
\langle\psi|\hat{x}^2|\psi\rangle &= \int_0^1 x^2|\psi(x)|^2 \,\mathrm{d}x \\
&= N^2 \int_0^1 x^4(1-x^2) \,\mathrm{d}x \\
&= N^2 \int_0^1 (x^4 - x^6) \,\mathrm{d}x \\
&= \frac{15}{2}\left(\frac{1}{5}-\frac{1}{7}\right) = \frac{3}{7}\,. \qquad \text{(2p)}
\end{aligned}
$$

Accordingly, the variance of the position is

$$
\sigma_x^2 = \langle\psi|\hat{x}^2|\psi\rangle - \langle\psi|\hat{x}|\psi\rangle^2 = \frac{3}{7} - \left(\frac{5}{8}\right)^2 = \frac{17}{448}\,. \qquad \text{(1p)}
$$

d) Expectation value of the momentum is obtained as

$$
\langle\psi|\hat{p}|\psi\rangle = \int_0^1 \overline{\psi(x)}(-i\hbar)\frac{\mathrm{d}\psi}{\mathrm{d}x}(x) \,\mathrm{d}x\,. \qquad \text{(1p)}
$$

For the derivative we get

$$
\frac{\mathrm{d}\psi}{\mathrm{d}x}(x) = N\frac{\mathrm{d}}{\mathrm{d}x}(x\sqrt{1-x^2}) = N\left(\sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}}\right)\,. \qquad \text{(1p)}
$$

Substituting this for the derivative, we get

$$
\begin{aligned}
\langle\psi|\hat{p}|\psi\rangle &= -i\hbar N^2 \int_0^1 \left(x(1-x^2) - x^3\right) \,\mathrm{d}x \\
&= -i\hbar N^2 \int_0^1 (x - 2x^3) \,\mathrm{d}x \\
&= -i\hbar N^2 (\frac{1}{2} - \frac{1}{2}) = 0\,. \qquad \text{(1p)}
\end{aligned}
$$

Problem 3

a) We must check that the given eigenstates satisfy the eigenvalue equation $\hat{H}|E\rangle = E|E\rangle$ with the given eigenvalues E. (1p) Let's first check the eigenstate $|E = 0\rangle$:

$$\begin{aligned}
\hat{H}|E=0\rangle &= \hat{H}\frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle) \\
&= \frac{1}{\sqrt{2}}\left[(|0\rangle + i|1\rangle) - i(-i|0\rangle + |1\rangle)\right] \\
&= \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle - |0\rangle - i|1\rangle) \\
&= 0 \\
&= 0|E=0\rangle\,. \qquad \text{(1p)}
\end{aligned}$$

Indeed, $|E = 0\rangle$ is an eigenstate of $\hat{H}$ with the eigenvalue 0.

Let's then check the other eigenstate $|E = 2\rangle$:

$$\begin{aligned}
\hat{H}|E=2\rangle &= \hat{H}\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) \\
&= \frac{1}{\sqrt{2}}\left[(|0\rangle + i|1\rangle) + i(-i|0\rangle + |1\rangle)\right] \\
&= \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle + |0\rangle + i|1\rangle) \\
&= 2\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) \\
&= 2|E=0\rangle\,. \qquad \text{(1p)}
\end{aligned}$$

Indeed, $|E = 2\rangle$ is an eigenstate of $\hat{H}$ with the eigenvalue 2.

b) The state vector at time t can be written in terms of the energy eigenstates and eigenvalues as

$$|\varphi(t)\rangle = \sum_E \langle E|\varphi(0)\rangle e^{-itE/\hbar}|E\rangle\,. \qquad \text{(1p)}$$

By substituting the eigenstates, eigenvalues and the initial state $|0\rangle$ we get

$$\begin{aligned}
|\varphi(t)\rangle &= \langle E=0|0\rangle e^{-it\cdot 0/\hbar}|E=0\rangle + \langle E=2|0\rangle e^{-it\cdot 2/\hbar}|E=2\rangle \qquad \text{(1p)} \\
&= \frac{1}{2}(|0\rangle - i|1\rangle) + \frac{e^{-2it/\hbar}}{2}(|0\rangle + i|1\rangle) \\
&= \frac{1}{2}(1 + e^{-2it/\hbar})|0\rangle - \frac{i}{2}(1 - e^{-2it/\hbar})|1\rangle\,. \qquad \text{(1p)}
\end{aligned}$$

c) The probability to measure the value of the qubit to be 0 at time t is obtained as $|\langle 0|\varphi(t)\rangle|^2$ (1p). The inner product gives $\langle 0|\varphi(t)\rangle = \frac{1}{2}(1 + e^{-2it/\hbar})$ (1p), so its norm squared is

$$\begin{aligned}
|\langle 0|\varphi(t)\rangle|^2 &= \overline{\langle 0|\varphi(t)\rangle}\langle 0|\varphi(t)\rangle \\
&= \frac{1}{4}(1 + e^{2it/\hbar})(1 + e^{-2it/\hbar}) \\
&= \frac{1}{4}(1 + e^{-2it/\hbar} + e^{2it/\hbar} + 1) \\
&= \frac{1}{2}(1 + \cos(2t/\hbar))\,. \qquad \text{(1p)}
\end{aligned}$$

d) When the qubit is measured and observed to be in the state $|0\rangle$ at time t, the quantum state of the qubit collapses to the state $|0\rangle$. Therefore, the qubit starts again its evolution from the state $|0\rangle$ at time t, and the probability to observe it to be in the state $|0\rangle$ at a later time $t + t'$ is given by $\frac{1}{2}(1 + \cos(2t'/\hbar))$. (1p)