ELEC-C9420 Introduction to Quantum Technology, Fall 20
Midterm exam 2, part B, 10.12.2020
teacher: Matti Raasakka

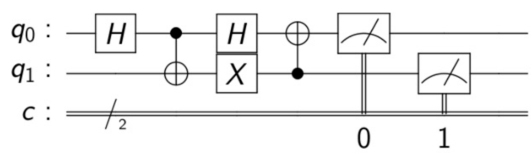
Problem B1

Consider a quantum particle in 1 spatial dimension described by the following wave function:

$$\phi(x) = \begin{cases} Nx(1-x^2) & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Here, $N = \sqrt{\frac{105}{8}}$ is the normalization constant. Show explicitly by computing the variances in position and momentum that this state of the particle satisfies the Heisenberg uncertainty principle.

Problem B2



The quantum circuit depicted above is applied to the initial state $|00\rangle$ of two qubits.

a) What is the state of the qubits just before the measurements?
b) Are the qubits entangled just before the measurements?
c) What is the probability to measure the first qubit q_0 to have value 0 in the first measurement?
d) After measuring the value of the first qubit, and finding it to be 0, what is the state of the two qubits just before the second measurement?

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Midterm exam 2, part B, model solutions
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Problem B1

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Here, $N = \sqrt{\frac{105}{8}}$ is the normalization constant. Show explicitly by computing the variances in position and momentum that this state of the particle satisfies the Heisenberg uncertainty principle.

Solution: The expectation value of position is given by

$$\langle \hat{x} \rangle = \int_0^1 x|\phi(x)|^2 \, \mathrm{d}x = N^2 \int_0^1 x^3(1-x^2)^2 \mathrm{d}x = \frac{105}{8}\frac{1}{24} = \frac{35}{64} \,. \quad (1\mathrm{p})$$

The expectation value of position squared is given by

$$\langle \hat{x}^2 \rangle = \int_0^1 x^2|\phi(x)|^2 \, \mathrm{d}x = N^2 \int_0^1 x^4(1-x^2)^2 \mathrm{d}x = \frac{105}{8}\frac{8}{315} = \frac{1}{3} \,. \quad (1\mathrm{p})$$

Therefore, we get for the variance in the position the value

$$\Delta x = \langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2 = \frac{1}{3} - \left(\frac{35}{64}\right)^2 = \frac{421}{12288} \,. \quad (1\mathrm{p})$$

For the expectation value of momentum we get

$$\begin{aligned} \langle \hat{p} \rangle &= \int_0^1 \overline{\phi(x)}(-i\hbar)\frac{\mathrm{d}\phi}{\mathrm{d}x}(x) \, \mathrm{d}x = -i\hbar N^2 \int_0^1 x(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}[x(1-x^2)] \, \mathrm{d}x \\ &= -i\hbar N^2 \underbrace{\int_0^1 x(1-x^2)(1-3x^2) \, \mathrm{d}x}_{=0} = 0 \,. \quad (1\mathrm{p}) \end{aligned}$$

For expectation value of position squared we get

$$\begin{aligned} \langle \hat{p}^2 \rangle &= \int_0^1 \overline{\phi(x)}(-i\hbar)^2\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}(x) \, \mathrm{d}x = -\hbar^2 N^2 \int_0^1 x(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2}[x(1-x^2)] \, \mathrm{d}x \\ &= -\hbar^2 N^2 \int_0^1 x(1-x^2)(-6x) \, \mathrm{d}x = -\hbar^2 \frac{105}{8}\left(-\frac{4}{5}\right) = \frac{21}{2}\hbar^2 \,. \quad (1\mathrm{p}) \end{aligned}$$

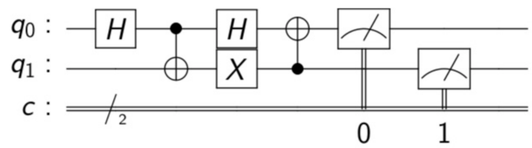
Accordingly, we get for the variance in the momentum

$$\Delta p = \langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2 = \frac{21}{2}\hbar^2 - 0 = \frac{21}{2}\hbar^2 \,. \quad (1\mathrm{p})$$

The Heisenberg uncertainty principle states that $\sigma_x \sigma_p \geq \frac{\hbar}{2}$, where $\sigma_x = \sqrt{\Delta x}$ is the standard deviation in position, and similarly for momentum. For the wavefunction $\phi(x)$ we have

$$\sigma_x \sigma_p = \sqrt{\frac{421}{12288}}\sqrt{\frac{21}{2}}\hbar = \sqrt{\frac{2947}{8192}}\hbar \,. \quad (1\mathrm{p})$$

Here, $\sqrt{\frac{2947}{8192}} = 0.59978\ldots > \frac{1}{2}$, so the state does satisfy the Heisenberg uncertainty principle. (1p)



Problem B2

The quantum circuit depicted above is applied to the initial state $|00\rangle$ of two qubits.

a) What is the state of the qubits just before the measurements?
b) Are the qubits entangled just before the measurements?
c) What is the probability to measure the first qubit q_0 to have value 0 in the first measurement?
d) After measuring the value of the first qubit, and finding it to be 0, what is the state of the two qubits just before the second measurement?

Solution: a) The state is mapped in the following way by the quantum gates before the measurements:

$$|00\rangle \overset{H\otimes I}{\mapsto} \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \overset{cX_{1,2}}{\mapsto} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \overset{I\otimes X}{\mapsto} \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$
$$\overset{H\otimes I}{\mapsto} \frac{1}{2}(|01\rangle + |11\rangle + |00\rangle - |10\rangle) \overset{cX_{2,1}}{\mapsto} \frac{1}{2}(|11\rangle + |01\rangle + |00\rangle - |10\rangle)$$

Thus, the state before the measurements is $|\Psi\rangle = \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle + |11\rangle)$. (2p)

b) The state $|\Psi\rangle$ cannot be expressed as a product state $|\phi\rangle|\psi\rangle$: The coefficients of the two states would have to satisfy

$$\phi_0\psi_0 = 1\,, \quad \phi_0\psi_1 = 1\,, \quad \phi_1\psi_0 = -1\,, \quad \phi_1\psi_1 = 1\,.$$

However, dividing the first equation by the second and the third by the fourth gives

$$\frac{\psi_0}{\psi_1} = 1\,, \quad \frac{\psi_0}{\psi_1} = -1\,,$$

which is a contradiction, and thus there are no such coefficients. Thus, the state $|\Psi\rangle$ is not a product state, and the qubits are entangled. (2p)

c) Each one of the basis states $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ has probability $\frac{1}{4}$. Accordingly, first qubit has probability $2 \cdot \frac{1}{4} = \frac{1}{2}$ to be observed to have value 0. (2p)

d) Due to the measurement, the state collapses to those branches of the superposition, which are compatible with the measurement result. Thus, after the measurement the state of the qubits is $|\Psi'\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$. (2p)