ELEC-E4130 Electromagnetic fields **Midterm 2, December 5, 2022**

Answer all questions on the provided separate answer sheets. No calculators or own papers are allowed. Submit at least one answer sheet with your name and student number. All answer sheets must be returned, including sketches and empty ones. You can keep this question sheet.

Question 1

a) Consider an air-filled rectangular waveguide with fixed dimensions. What happens to the wave impedance of the TE_{10} mode when the operational frequency is increased significantly above TE_{10} cutoff frequency?

b) What is the phase and group velocity of a lossless coax line, operating in a TEM mode, with vacuum between the conductors. Now fill the coax line with a dielectric with refractive index n = 2. What is the new phase and group velocities. How does the wave impedance vary as a function of frequency for both cases?

c) Define radiation pattern and sidelobe level.

d) Define radiation resistance of an antenna.

Question 2

a) Consider two TE (perpendicular) polarized waves in air incident on a dielectric film of thickness ℓ_1 and refractive index n_1. Wave A at frequency f_A is incident normally and Wave B at frequency f_B is incident at angle θ_a. What is the relation between f_A and f_B such that the tangential (to the planar interfaces) components of each wave experience the same phase shift as it bounces around inside the film? Write in terms of θ_a and n_1

b) Now consider wave B with a 45-degree incidence angle and assume that the refractive index of the film is $n_1 = \sqrt{2}$. Use what you know about constructive/destructive interference at the film interface between the initial reflection and the subsequent paths through the film to design a film thickness such that the first reflection coefficient maximum is f_0. Write your answer in terms of f_0 and c_0

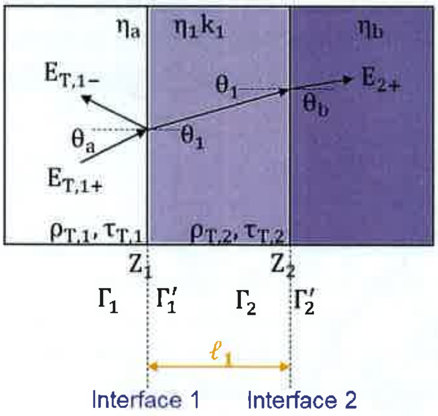
Question 3

a) Consider an a = 2b rectangular waveguide "filled" with vacuum. Find dimensions a and b where the lowest order mode that propagates with wave impedance of 1 Ω at a frequency f = c/2π Hz. Write your answers in terms of characteristic impedance η.

b) Consider the same waveguide as part a). What is the ratio of power dissipated per unit length in the TE_{01} mode compared to the power dissipated per unit length in the TE_{10} (lowest order mode)? The fields for both modes are given in the formula sheet. Assume that the fields have the same amplitude coefficient ($A_{01} = A_{10}$) and that the current density is a surface current density thus the loss per unit area and current density are:

$$P_\ell = \frac{R_S}{2} \int_A |\mathbf{J}_S|^2 ds \qquad \mathbf{J}_S = \mathbf{a}_n \times \mathbf{H}_1$$

Multilayer equations



$$\Gamma_{T,m} = \frac{\rho_{T,m} + \Gamma_{T,m+1} e^{-j2\delta_m}}{1 + \rho_{T,m}\Gamma_{T,m+1} e^{-j2\delta_m}}$$

$$\delta_m = k_m \ell_m \cos(\theta_m)$$

$$\Gamma_1 = \text{goal coefficient}$$

$$\Gamma_{T,M+1} = \rho_{T,m+1}$$

$$\eta_T = \begin{cases} \eta\cos(\theta) & \parallel, \text{TM} \\ \dfrac{\eta}{\cos(\theta)} & \perp, \text{TE} \end{cases}$$

$$\rho_{T,1} = \frac{\eta_{T,1} - \eta_{T,a}}{\eta_1 + \eta_a}$$

$$\tau_{T,1} = 1 + \rho_{T,1}$$

$$\rho_{T,2} = \frac{\eta_{T,b} - \eta_{T,1}}{\eta_{T,b} + \eta_{T,1}}$$

$$\tau_{T,2} = 1 + \rho_{T,2}$$

$$\tau_{T,1}\tau'_{T,1} = 1 - \rho_{T,1}^2$$

Waveguide Equations

TM modes

$$E_z(x,y,z) = A_{mn}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta z}$$

$$H_x(x,y,z) = \frac{j\omega\varepsilon n\pi}{bk_c^2}A_{mn}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$$

$$H_y(x,y,z) = \frac{-j\omega\varepsilon m\pi}{ak_c^2}A_{mn}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$$

$$E_x(x,y,z) = \frac{-j\beta m\pi}{ak_c^2}A_{mn}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$$

$$E_y(x,y,z) = \frac{-j\beta n\pi}{bk_c^2}A_{mn}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$$

$$Z_{TM} = \frac{E_x}{H_y} = \frac{-E_y}{H_x} = \frac{\beta}{\omega\varepsilon} = \frac{\beta}{k}\eta$$

TE modes

$$H_z(x,y,z) = B_{mn}\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta z}$$

$$E_x(x,y,z) = \frac{j\omega\mu n\pi}{bk_c^2}B_{mn}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$$

$$E_y(x,y,z) = \frac{-j\omega\mu m\pi}{ak_c^2}B_{mn}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$$

$$H_x(x,y,z) = \frac{j\beta m\pi}{ak_c^2}B_{mn}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$$

$$H_y(x,y,z) = \frac{j\beta n\pi}{bk_c^2}B_{mn}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$$

$$Z_{TE} = \frac{E_x}{H_y} = \frac{-E_y}{H_x} = \frac{\omega\mu}{\beta} = \frac{k}{\beta}\eta$$

$$k_c = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

$$\beta = \sqrt{k^2 - k_c^2}$$

TE_{10} Fields

$$H_z = A_{10}\cos\left(\frac{\pi}{a}x\right)e^{-j\beta_{10}z}$$

$$E_x = 0$$

$$E_y = -j\frac{\omega\mu a}{\pi}A_{10}\sin\left(\frac{\pi}{a}x\right)e^{-j\beta_{10}z}$$

$$H_x = j\frac{\beta_{10}a}{\pi}A_{10}\sin\left(\frac{\pi}{a}x\right)e^{-j\beta_{10}z}$$

$$H_y = 0$$

$$P_{\ell,10} = R_s A_{10}^2\left[\frac{a}{2} + b + \frac{a^3\beta_{10}^2}{2\pi^2}\right]$$

TE_{01} Fields

$$H_z = A_{01}\cos\left(\frac{\pi}{b}y\right)e^{-j\beta_{01}z}$$

$$E_x = j\frac{\omega\mu b}{\pi}A_{01}\sin\left(\frac{\pi}{b}y\right)e^{-j\beta_{01}z}$$

$$E_y = 0$$

$$H_x = 0$$

$$H_y = j\frac{\beta_{01}b}{\pi}A_{01}\sin\left(\frac{\pi}{b}y\right)e^{-j\beta_{01}z}$$

$$P_{\ell,01} = ?$$

Some trig identities

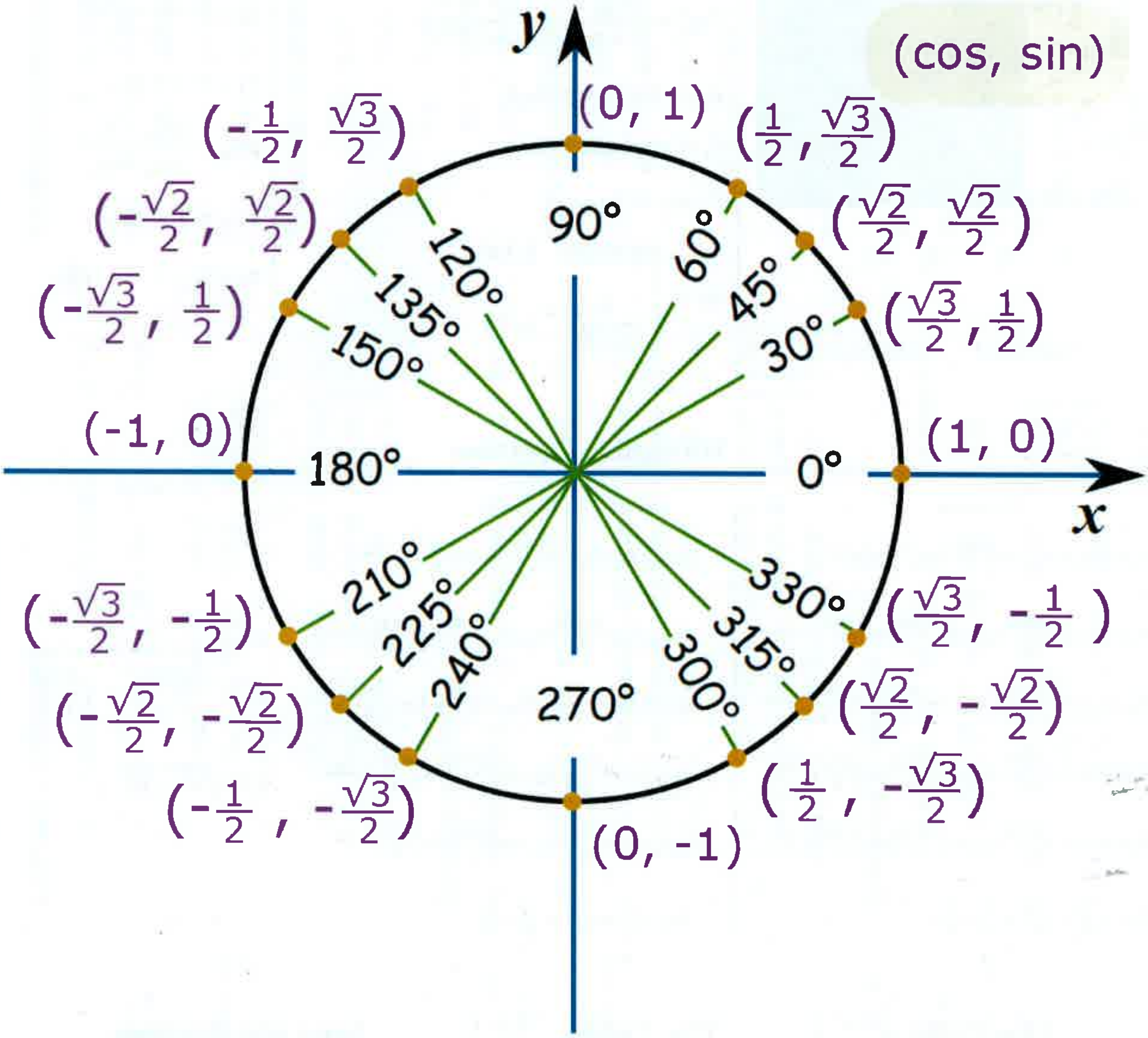
$$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + C$$

$$\int \sin^2(ax)\,dx = \frac{x}{2} - \frac{1}{2a}\sin(ax)\cos(ax) + C$$

$$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + C$$

$$\int \cos^2(ax)\,dx = \frac{x}{2} + \frac{1}{2a}\sin(ax)\cos(ax) + C$$

Unit Circle



y
x
(cos, sin)
(0, 1)
(1/2, √3/2)
(√2/2, √2/2)
(√3/2, 1/2)
(1, 0)
(√3/2, -1/2)
(√2/2, -√2/2)
(1/2, -√3/2)
(0, -1)
(-1/2, -√3/2)
(-√2/2, -√2/2)
(-√3/2, -1/2)
(-1, 0)
(-√3/2, 1/2)
(-√2/2, √2/2)
(-1/2, √3/2)
0°
30°
45°
60°
90°
120°
135°
150°
180°
210°
225°
240°
270°
300°
315°
330°

Nabla operations

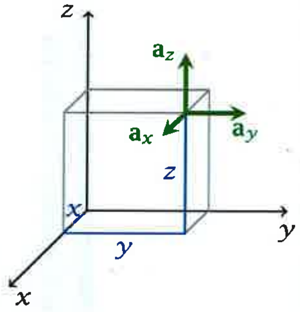
Cartesian coordinates (x, y, z)

$$\nabla V = \mathbf{a}_x \frac{\partial V}{\partial x} + \mathbf{a}_y \frac{\partial V}{\partial y} + \mathbf{a}_z \frac{\partial V}{\partial z}$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$



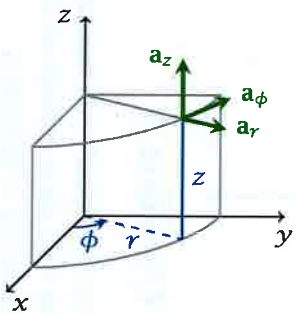
Cylindrical coordinates (r, ϕ, z)

$$\nabla V = \mathbf{a}_r \frac{\partial V}{\partial r} + \mathbf{a}_\phi \frac{1}{r}\frac{\partial V}{\partial \phi} + \mathbf{a}_z \frac{\partial V}{\partial z}$$

$$\nabla \times \mathbf{A} = \frac{1}{r} \begin{vmatrix} \mathbf{a}_r & \mathbf{a}_\phi r & \mathbf{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}\left(rA_r\right) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$



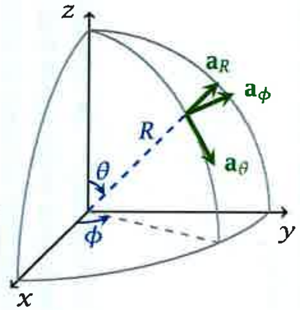
Spherical coordinates (R, θ, ϕ)

$$\nabla V = \mathbf{a}_R \frac{\partial V}{\partial R} + \mathbf{a}_\theta \frac{1}{R}\frac{\partial V}{\partial \theta} + \mathbf{a}_\phi \frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \frac{1}{R^2 \sin\theta} \begin{vmatrix} \mathbf{a}_R & \mathbf{a}_\theta R & \mathbf{a}_\phi R\sin\theta \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_R & RA_\theta & (R\sin\theta)\,A_\phi \end{vmatrix}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 A_R\right) + \frac{1}{R\sin\theta}\frac{\partial}{\partial \theta}\left(A_\theta \sin\theta\right) + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$



Coordinate transformations

Cartesian ↔ Cylindrical

$x = r\cos\phi, \quad y = r\sin\phi, \quad z = z$

$r = \sqrt{x^2 + y^2}, \quad \phi = \tan^{-1}\frac{y}{x}, \quad z = z$

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_r \\ A_\phi \\ A_z \end{pmatrix}$$

$$\begin{pmatrix} A_r \\ A_\phi \\ A_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

Cartesian ↔ Spherical

$x = R\sin\theta\cos\phi, \quad y = R\sin\theta\sin\phi, \quad z = R\cos\theta$

$R = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}, \quad \phi = \tan^{-1}\frac{y}{x}$

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} A_R \\ A_\theta \\ A_\phi \end{pmatrix}$$

$$\begin{pmatrix} A_R \\ A_\theta \\ A_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

Cylindrical ↔ Spherical

$r = R\sin\theta, \quad \phi = \phi, \quad z = R\cos\theta$

$R = \sqrt{r^2 + z^2}, \quad \theta = \tan^{-1}\frac{r}{z}, \quad \phi = \phi$

$$\begin{pmatrix} A_r \\ A_\phi \\ A_z \end{pmatrix} = \begin{pmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} A_R \\ A_\theta \\ A_\phi \end{pmatrix}$$

$$\begin{pmatrix} A_R \\ A_\theta \\ A_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} A_r \\ A_\phi \\ A_z \end{pmatrix}$$

Other useful formulas

Cartesian coordinates

$$d\boldsymbol{\ell} = \mathbf{a}_x\,dx + \mathbf{a}_y\,dy + \mathbf{a}_z\,dz$$

$ds_x = dy\,dz$

$ds_y = dx\,dz$

$ds_z = dx\,dy$

$dv = dx\,dy\,dz$

Cylindrical coordinates

$$d\boldsymbol{\ell} = \mathbf{a}_r\,dr + \mathbf{a}_\phi r\,d\phi + \mathbf{a}_z\,dz$$

$ds_r = r\,d\phi\,dz$

$ds_\phi = dr\,dz$

$ds_z = r\,dr\,d\phi$

$dv = r\,dr\,d\phi\,dz$

Spherical coordinates

$$d\boldsymbol{\ell} = \mathbf{a}_R\,dR + \mathbf{a}_\theta R\,d\theta + \mathbf{a}_\phi R\sin\theta\,d\phi$$

$ds_R = R^2\sin\theta\,d\theta\,d\phi$

$ds_\theta = R\sin\theta\,dR\,d\phi$

$ds_\phi = R\,dR\,d\theta$

$dv = R^2\sin\theta\,dR\,d\theta\,d\phi$

Divergence theorem $\int_V \nabla\cdot\mathbf{A}\,dv = \oint_S \mathbf{A}\cdot d\mathbf{s}$

Stokes' theorem $\int_S (\nabla\times\mathbf{A})\cdot d\mathbf{s} = \oint_C \mathbf{A}\cdot d\boldsymbol{\ell}$

Constants

$c = 299\,792\,458\ \frac{\mathrm{m}}{\mathrm{s}}$

$\mu_0 = 4\pi\times10^{-7}\ \frac{\mathrm{Vs}}{\mathrm{Am}} \approx 1.257\times10^{-6}\ \frac{\mathrm{H}}{\mathrm{m}}$

$\varepsilon_0 = \frac{1}{\mu_0 c^2} \approx 8.854\times10^{-12}\ \frac{\mathrm{As}}{\mathrm{Vm}} \quad \left(=\frac{\mathrm{F}}{\mathrm{m}}\right)$

$e \approx 1.602\times10^{-19}\ \mathrm{C}$