Question 1

a) Consider an air-filled rectangular waveguide with fixed dimensions. What happens to the wave impedance of the TE_{10} mode when the operational frequency is increased significantly above TE_{10} cutoff frequency?

The wave impedance converges to the characteristic impedance of the waveguide fill.

b) What is the phase and group velocity of a lossless coax line, operating in a TEM mode, with vacuum between the conductors. Now fill the coax line with a dielectric with refractive index n = 2. What is the new phase and group velocities. How does the wave impedance vary as a function of frequency for both cases?

$$u_p(n = 1) = c_0$$

$$u_g(n = 1) = c_0$$

$$u_p(n = 2) = \frac{c_0}{2}$$

$$u_g(n = 2) = \frac{c_0}{2}$$

Wave impedance for TEM propagation is invariant to frequency if refractive index is also invariant to frequency

c) Define radiation pattern and sidelobe level.

The radiation pattern of an antenna is the relative field strength as function of direction at a fixed distance in the far field. The radiation pattern is normalized so that its maximum amplitude is 1 in the direction of the main beam. The smaller local maxima of the radiation pattern are called side lobes and the amplitude of the largest sidelobe is called sidelobe level.

d) Define radiation resistance of an antenna.

The radiation resistance is an equivalent resistance that would dissipate the same amount of power as the antenna radiates when the antenna feed current flows through this hypothetical resistor.

Question 2

a) Consider two TE (perpendicular) polarized waves in air incident on a dielectric film of thickness ℓ_1 and refractive index n_1. Wave A at frequency f_A is incident normally and Wave B at frequency f_B is incident at angle θ_a. What is the relation between f_A and f_B such that the tangential (to the planar interfaces) components of each wave experience the same phase shift as it bounces around inside the film? Write in terms of θ_a and n_1

Equal phase after one round trip

The normally incident field is tangential to the interface. The phase complex exponential of the first two-pass path for oblique incidence describes the evolution of the phase of the tangential field component

$$e^{-j2k_A\ell_1} = e^{-j2k_B\ell_1\cos(\theta_1)}$$

$$k_A\ell_1 = k_B\ell_1\cos(\theta_1)$$

$$\frac{2\pi n_1}{\lambda_A} = \frac{2\pi n_1}{\lambda_B}\cos(\theta_1)$$

$$\lambda_A = \frac{\lambda_B}{\cos(\theta_1)}$$

$$f_A = f_B\cos(\theta_1)$$

$$n_a\sin(\theta_a) = n_1\sin(\theta_1)$$

$$\theta_1 = \sin^{-1}\left(\frac{n_a}{n_1}\sin(\theta_a)\right) = \sin^{-1}\left(\frac{1}{n_1}\sin(\theta_a)\right)$$

$$f_A = f_B\cos\left(\sin^{-1}\left(\frac{1}{n_1}\sin(\theta_a)\right)\right)$$

b) Now consider the 45-degree incidence angle and assume that the refractive index of the film is $n = \sqrt{2}$. Use what you know about constructive/destructive interference at the film interface between the initial reflection and the subsequent paths through the film to design a film thickness such that the first reflection coefficient maximum is f_0. Write your answer in terms of f_0 and c_0

Angle inside the film

$$n_a\sin(\theta_a) = n_1\sin(\theta_1)$$

$$\theta_1 = \sin^{-1}\left(\frac{n_a}{n_1}\sin(\theta_a)\right) = \sin^{-1}\left(\frac{1}{\sqrt{2}}\sin\left(\frac{\pi}{4}\right)\right) = \sin^{-1}\left(\frac{1}{\sqrt{2}}\frac{\sqrt{2}}{2}\right) = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

Fresnel coefficients

$$\rho_1 = \frac{\frac{\eta_0}{n_1\cos(\theta_1)} - \frac{\eta_0}{n_a\cos(\theta_a)}}{\frac{\eta_0}{n_1\cos(\theta_1)} + \frac{\eta_0}{n_a\cos(\theta_a)}} = \frac{\frac{1}{\sqrt{2}\cos\left(\frac{\pi}{6}\right)} - \frac{1}{\cos\left(\frac{\pi}{4}\right)}}{\frac{1}{\sqrt{2}\cos\left(\frac{\pi}{6}\right)} + \frac{1}{\cos\left(\frac{\pi}{4}\right)}} = \frac{\frac{1}{\sqrt{2}\frac{\sqrt{3}}{2}} - \frac{1}{\frac{\sqrt{2}}{2}}}{\frac{1}{\sqrt{2}\frac{\sqrt{3}}{2}} + \frac{1}{\frac{\sqrt{2}}{2}}} = \frac{\frac{1}{\sqrt{3}} - 1}{\frac{1}{\sqrt{3}} + 1} = \frac{1-\sqrt{3}}{1+\sqrt{3}} = |\rho_1|e^{-j\pi}$$

$$\rho_2 = -\rho_1 = |\rho_2|e^{-j0}$$

$$\tau_1 = 1 + \rho_1 = |\tau_1|e^{-j0}$$

$$\tau_2 = 1 + \rho_2 = |\tau_2|e^{-j0}$$

$$\tau_1\tau_1' = 1 - \rho_1^2 = |\tau_1\tau_1'|e^{-j0}$$

Note that it is not necessary to actually compute the Fresnel coefficients, its only necessary to confirm the coefficients' signs; positive or negative.

Initial reflection:

$$\rho_1 E_0 = |\rho_1|e^{-j\pi}E_0$$

First path through the film

$$\tau_1\tau_1'\rho_2 e^{-j2k_1\ell_1\cos(\theta_1)}E_0 = |\tau_1\tau_1'||\rho_2|e^{-j2k_1\ell_1\cos(\theta_1)}E_0$$

Constructive interference

To maximize reflection the reverse traveling waves must add constructively therefore their phases must add constructively

$$e^{-j\pi} = e^{-j2k_1\ell_1\cos(\theta_1)}$$

$$\pi = 2k_1\ell_1\cos(\theta_1)$$

$$\pi = 2\frac{2\pi n_1}{\lambda_0}\ell_1\cos(\theta_1)$$

$$1 = \frac{4n_1}{\lambda_0}\ell_1\cos(\theta_1)$$

$$\ell_1 = \frac{\lambda_0}{4n_1\cos(\theta_1)} = \frac{c_0}{4f_0n_1\cos(\theta_1)} = \frac{c_0}{4f_0\sqrt{2}\frac{\sqrt{3}}{2}}$$

$$\ell_1 = \frac{c_0}{2\sqrt{6}f_0}$$

Question 3

a) Consider an $a = 2b$ rectangular waveguide "filled" with vacuum. Find dimensions a and b where the lowest order mode that propagates with wave impedance of 1 Ω at a frequency $f = c/2\pi$ Hz. Write your answers in terms of characteristic impedance η.

TE waves propagate with wave impedance in the range of (η, ∞). TM waves propagate with wave impedance in the range of (0, η). The inside of the wavguide is vacuum so η ~ 377 Ω. Therefore, to reach 1 Ω we must operate in a TM mode, very very close to cutoff. The lowest order TM mode is TM_{11}.

Propagation constant

$$\beta = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

$$\beta_{11} = \sqrt{k^2 - \left(\frac{\pi}{a}\right)^2 - \left(\frac{\pi}{b}\right)^2} = \sqrt{k^2 - \left(\frac{\pi}{2b}\right)^2 - \left(\frac{\pi}{b}\right)^2} = \sqrt{k^2 - \frac{5}{4}\left(\frac{\pi}{b}\right)^2}$$

$$f = \frac{c}{2\pi} \rightarrow \lambda = \frac{c}{f} = 2\pi$$

$$k = \frac{2\pi}{\lambda} = 1$$

$$\beta_{11} = \sqrt{1 - \frac{5}{4}\left(\frac{\pi}{b}\right)^2}$$

TM Wave Impedance

$$Z_{TM} = \frac{\beta}{k}\eta = 1$$

$$\eta\sqrt{1 - \frac{5}{4}\left(\frac{\pi}{b}\right)^2} = 1$$

$$1 - \frac{5}{4}\left(\frac{\pi}{b}\right)^2 = \left(\frac{1}{\eta}\right)^2$$

$$\frac{5}{4}\left(\frac{\pi}{b}\right)^2 = 1 - \left(\frac{1}{\eta}\right)^2$$

$$\left(\frac{\pi}{b}\right)^2 = \frac{4}{5}\left(1 - \left(\frac{1}{\eta}\right)^2\right) = \frac{4}{5}\left(\frac{\eta^2 - 1}{\eta^2}\right)$$

$$b = \pi\sqrt{\frac{5}{4}\left(\frac{\eta^2}{\eta^2 - 1}\right)} = \frac{\pi}{2}\sqrt{5\left(\frac{\eta^2}{\eta^2 - 1}\right)}$$

$$b = \frac{\pi}{2}\sqrt{5\left(\frac{\eta^2}{\eta^2 - 1}\right)}$$

$$a = \pi\sqrt{5\left(\frac{\eta^2}{\eta^2 - 1}\right)}$$

b) Consider the same waveguide as part a). What is the ratio of power dissipated per unit length in the TE_{01} mode compared to the power dissipated per unit length in the TE_{10} (lowest order mode)? The fields for both modes are given in the formula sheet. Assume that the fields have the same amplitude coefficient ($A_{01} = A_{10}$) and that the current density is a surface current density thus the loss per unit area and current density are:

$$P_\ell = \frac{R_s}{2}\int_A |\mathbf{J_s}|^2 ds$$
$$\mathbf{J}_s = \mathbf{a_n} \times \mathbf{H_1}$$

The TE10 and TE01 fields listed in the equation sheet are listed below: Note that the TE10 computations are included as a reference. If you look at the fields you can see their functional form is related by a substitution of (a,x) for (b,y) which means the result of the power dissipated per unit length equation can be obtained by a simple substitution of a, b, and β: Assume the functional form of power dissipated as:

$$P_{\ell,10}(a, b, \beta_{10})$$
$$P_{\ell,01}(a, b, \beta_{01})$$

Then the solution can be found as:

$$P_{\ell,01}(a, b, \beta_{01}) = P_{\ell,10}(a \to b, b \to a, \beta_{10} \to \beta_{01})$$

This can also be calculated directly:

TE_{10}	TE_{01}
$H_z = A_{10}\cos\left(\frac{\pi}{a}x\right)e^{-j\beta_{10}z}$	$H_z = A_{01}\cos\left(\frac{\pi}{b}y\right)e^{-j\beta_{01}z}$
$E_x = 0$	$E_x = j\frac{\omega\mu b}{\pi}A_{01}\sin\left(\frac{\pi}{b}y\right)e^{-j\beta_{01}z}$
$E_y = -j\frac{\omega\mu a}{\pi}A_{10}\sin\left(\frac{\pi}{a}x\right)e^{-j\beta_{10}z}$	$E_y = 0$
$H_x = j\frac{\beta_{10}a}{\pi}A_{10}\sin\left(\frac{\pi}{a}x\right)e^{-j\beta_{10}z}$	$H_x = 0$
$H_y = 0$	$H_y = j\frac{\beta_{01}b}{\pi}A_{01}\sin\left(\frac{\pi}{b}y\right)e^{-j\beta_{01}z}$
$P_{\ell,10} = R_sA_{10}^2\left[\frac{a}{2} + b + \frac{a^3\beta_{10}^2}{2\pi^2}\right]$	$P_{\ell,01} = ?$

TE_{10} Current Density (For reference)	TE_{01} Current Density
Floor, y = 0 $\mathbf{J}_s = \mathbf{a_n} \times \mathbf{H_1} = \mathbf{a_y} \times (\mathbf{a_z}H_z + \mathbf{a_x}H_x)$ $\mathbf{J}_s = \left(+\mathbf{a_x}A_{10}\cos\left(\frac{\pi}{a}x\right) - \mathbf{a_z}A_{10}\frac{j\beta_{10}a}{\pi}\sin\left(\frac{\pi}{a}x\right)\right)e^{-j\beta_{10}z}$	**Floor, y = 0** $\mathbf{J}_s = \mathbf{a_n} \times \mathbf{H_1} = \mathbf{a_y} \times \mathbf{a_z}H_z$ $\mathbf{J}_s = \mathbf{a_x}A_{01}e^{-j\beta_{01}z}$
Ceiling, y = b $\mathbf{J}_s = \mathbf{a_n} \times \mathbf{H_1} = -\mathbf{a_y} \times (\mathbf{a_z}H_z + \mathbf{a_x}H_x)$ $\mathbf{J}_s = \left(-\mathbf{a_x}A_{10}\cos\left(\frac{\pi}{a}x\right) + \mathbf{a_z}A_{10}\frac{j\beta_{10}a}{\pi}\sin\left(\frac{\pi}{a}x\right)\right)e^{-j\beta_{10}z}$	**Ceiling, y = b** $\mathbf{J}_s = \mathbf{a_n} \times \mathbf{H_1} = -\mathbf{a_y} \times \mathbf{a_z}H_z$ $\mathbf{J}_s = \mathbf{a_x}A_{01}e^{-j\beta_{01}z}$
Left Wall, x = 0 $\mathbf{J_s} = \mathbf{a_n} \times \mathbf{H_1} = \mathbf{a_x} \times \mathbf{a_z}H_z$ $\mathbf{J}_s = -\mathbf{a_y}A_{10}e^{-j\beta_{10}z}$	**Left Wall, x = 0** $\mathbf{J_s} = \mathbf{a_n} \times \mathbf{H_1} = \mathbf{a_x} \times (\mathbf{a_z}H_z + \mathbf{a_y}H_y)$ $\mathbf{J}_s = \left(-\mathbf{a_y}A_{01}\cos\left(\frac{\pi}{b}y\right) + \mathbf{a_z}A_{01}\frac{j\beta_{01}b}{\pi}\sin\left(\frac{\pi}{b}y\right)\right)e^{-j\beta_{01}z}$
Right Wall, x = a $\mathbf{J_s} = \mathbf{a_n} \times \mathbf{H_1} = -\mathbf{a_x} \times \mathbf{a_z}H_z$ $\mathbf{J}_s = -\mathbf{a_y}A_{10}e^{-j\beta_{10}z}$	**Right Wall, x = a** $\mathbf{J_s} = \mathbf{a_n} \times \mathbf{H_1} = -\mathbf{a_x} \times \mathbf{a_z}H_z$ $\mathbf{J}_s = \left(+\mathbf{a_y}A_{01}\cos\left(\frac{\pi}{b}y\right) - \mathbf{a_z}A_{01}\frac{j\beta_{01}b}{\pi}\sin\left(\frac{\pi}{b}y\right)\right)e^{-j\beta_{01}z}$

TE$_{10}$ Power Dissipated	TE$_{01}$ Current Density
Floor, y = 0 $P_\ell = \frac{R_s}{2}\int_{x=0}^{a}\left[\lvert\mathbf{J}_{s,x}\rvert^2 + \lvert\mathbf{J}_{s,z}\rvert^2\right]dx$ $P_\ell = \frac{R_s}{2}A_{10}^2\left[\int_{x=0}^{a}\cos^2\left(\frac{\pi}{a}x\right)dx + \frac{a^2\beta_{10}^2}{\pi^2}\int_{x=0}^{a}\sin^2\left(\frac{\pi}{a}x\right)dx\right]$ $P_\ell = \frac{R_s}{2}A_{10}^2\left[\frac{a}{2}+\frac{a^2\beta_{10}^2}{\pi^2}\frac{a}{2}\right]$	**Floor, y = 0** $P_\ell = \frac{R_s}{2}\int_{x=0}^{a}\lvert\mathbf{J}_{s,x}\rvert^2 dx$ $P_\ell = \frac{R_s}{2}A_{01}^2\int_{x=0}^{a}dx$ $P_\ell = \frac{R_s}{2}A_{01}^2 a$
Ceiling, y = b $P_\ell = \frac{R_s}{2}\int_{x=0}^{a}\left[\lvert\mathbf{J}_{s,x}\rvert^2 + \lvert\mathbf{J}_{s,z}\rvert^2\right]dx$ $P_\ell = \frac{R_s}{2}A_{10}^2\left[\int_{x=0}^{a}\cos^2\left(\frac{\pi}{a}x\right)dx + \frac{a^2\beta_{10}^2}{\pi^2}\int_{x=0}^{a}\sin^2\left(\frac{\pi}{a}x\right)dx\right]$ $P_\ell = \frac{R_s}{2}A_{10}^2\left[\frac{a}{2}+\frac{a^2\beta_{10}^2}{\pi^2}\frac{a}{2}\right]$	**Ceiling, y = b** $P_\ell = \frac{R_s}{2}\int_{x=0}^{a}\lvert\mathbf{J}_{s,x}\rvert^2 dx$ $P_\ell = \frac{R_s}{2}A_{01}^2\int_{x=0}^{a}dx$ $P_\ell = \frac{R_s}{2}A_{01}^2 a$
Left Wall, x = 0 $P_\ell = \frac{R_s}{2}\int_{y=0}^{b}\lvert\mathbf{J}_{s,y}\rvert^2 dy$ $P_\ell = \frac{R_s}{2}A_{10}^2\int_{y=0}^{b}dy$ $P_\ell = \frac{R_s}{2}A_{10}^2 b$	**Left Wall, x = 0** $P_\ell = \frac{R_s}{2}\int_{y=0}^{b}\left[\lvert\mathbf{J}_{s,y}\rvert^2 + \lvert\mathbf{J}_{s,z}\rvert^2\right]dx$ $P_\ell = \frac{R_s}{2}A_{01}^2\left[\int_{y=0}^{b}\cos^2\left(\frac{\pi}{b}y\right)dx + \frac{b^2\beta_{01}^2}{\pi^2}\int_{y=0}^{b}\sin^2\left(\frac{\pi}{b}y\right)dx\right]$ $P_\ell = \frac{R_s}{2}A_{01}^2\left[\frac{b}{2}+\frac{b^2\beta_{01}^2}{\pi^2}\frac{b}{2}\right]$
Right Wall, x = a $P_\ell = \frac{R_s}{2}\int_{y=0}^{b}\lvert\mathbf{J}_{s,y}\rvert^2 dy$ $P_\ell = \frac{R_s}{2}A_{10}^2\int_{y=0}^{b}dy$ $P_\ell = \frac{R_s}{2}A_{10}^2 b$	**Right Wall, x = a** $P_\ell = \frac{R_s}{2}\int_{y=0}^{b}\left[\lvert\mathbf{J}_{s,y}\rvert^2 + \lvert\mathbf{J}_{s,z}\rvert^2\right]dx$ $P_\ell = \frac{R_s}{2}A_{01}^2\left[\int_{y=0}^{b}\cos^2\left(\frac{\pi}{b}y\right)dx + \frac{b^2\beta_{01}^2}{\pi^2}\int_{y=0}^{b}\sin^2\left(\frac{\pi}{b}y\right)dx\right]$ $P_\ell = \frac{R_s}{2}A_{01}^2\left[\frac{b}{2}+\frac{b^2\beta_{01}^2}{\pi^2}\frac{b}{2}\right]$

TE$_{10}$ TOTAL Power Dissipated	TE$_{01}$ TOTAL Current Density
$P_{\ell,10} = \frac{R_s}{2}A_{10}^2\left[\frac{a}{2}+\frac{a^2\beta_{10}^2}{\pi^2}\frac{a}{2}+\frac{a}{2}+\frac{a^2\beta_{10}^2}{\pi^2}\frac{a}{2}+b+b\right]$ $P_{\ell,10} = R_sA_{10}^2\left[\frac{a}{2}+b+\frac{a^3\beta_{10}^2}{2\pi^2}\right]$ $P_{\ell,10} = \frac{R_sA_{10}^2}{2\pi^2}\left[\pi^2a+2\pi^2b+a^3\beta_{10}^2\right]$	$P_{\ell,01} = \frac{R_s}{2}A_{01}^2\left[\frac{b}{2}+\frac{b^2\beta_{01}^2}{\pi^2}\frac{b}{2}+\frac{b}{2}+\frac{b^2\beta_{01}^2}{\pi^2}\frac{b}{2}+a+a\right]$ $P_{\ell,01} = R_sA_{01}^2\left[\frac{b}{2}+a+\frac{b^3\beta_{01}^2}{2\pi^2}\right]$ $P_{\ell,01} = \frac{R_sA_{01}^2}{2\pi^2}\left[\pi^2b+2\pi^2a+b^3\beta_{01}^2\right]$

TE_{10} TOTAL Power Dissipated in terms of b	TE_{01} TOTAL Current Density in terms of b
$P_{\ell,10} = \frac{R_s A_{10}^2}{2\pi^2}[2\pi^2 b + 2\pi^2 b + 8b^3\beta_{10}^2]$ $P_{\ell,10} = \frac{R_s A_{10}^2 b}{2\pi^2}[2\pi^2 + 2\pi^2 + 8b^2\beta_{10}^2]$ $P_{\ell,10} = \frac{R_s A_{10}^2 b}{2\pi^2}[4\pi^2 + 8b^2\beta_{10}^2]$	$P_{\ell,01} = \frac{R_s A_{01}^2}{2\pi^2}[\pi^2 b + 4\pi^2 b + b^3\beta_{01}^2]$ $P_{\ell,01} = \frac{R_s A_{01}^2 b}{2\pi^2}[\pi^2 + 4\pi^2 + b^2\beta_{01}^2]$ $P_{\ell,01} = \frac{R_s A_{01}^2 b}{2\pi^2}[5\pi^2 + b^2\beta_{01}^2]$

Ratio of power lost in the modes

$$\frac{P_{\ell,01}}{P_{\ell,10}} = \frac{5\pi^2 + b^2\beta_{01}^2}{4\pi^2 + 8b^2\beta_{10}^2}$$

$$\beta_{01} = \sqrt{k^2 - \left(\frac{\pi}{b}\right)^2} = \sqrt{\frac{b^2k^2 - \pi^2}{b^2}} = \frac{1}{b}\sqrt{b^2k^2 - \pi^2}$$

$$\beta_{10} = \sqrt{k^2 - \left(\frac{\pi}{a}\right)^2} = \sqrt{k^2 - \left(\frac{\pi}{2b}\right)^2} = \frac{1}{b}\sqrt{b^2k^2 - \pi^2/4}$$

$$\frac{P_{\ell,01}}{P_{\ell,10}} = \frac{5\pi^2 + b^2(b^2k^2 - \pi^2)}{4\pi^2 + 8b^2(b^2k^2 - \pi^2/4)}$$