PHYS-C0252 — Quantum Mechanics

Final examination

Tuesday, 14 December, 09:00–12.00

Instructions

- Your answers should be legible and appropriately numbered.
- Your answers should contain relevant intermediate steps and explanations for the calculations.
- No calculators, cheat sheets or other materials are allowed.
- Return the exam paper when you are finished.

1. The Hamiltonian of a quantum harmonic oscillator in terms of the position operator $\hat{x}$ and momentum operator $\hat{p}$ is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2.$$

By choosing $\hat{P} = \hat{p}/\sqrt{m\omega\hbar}$ and $\hat{Q} = \hat{x}\sqrt{m\omega/\hbar}$, the Hamiltonian may be rewritten as

$$\hat{H} = \frac{\hbar\omega}{2}\left(\hat{Q}^2 + \hat{P}^2\right).$$

$\omega = \frac{2\pi}{T}$

$\hbar = \frac{h}{2\pi} \cdot \frac{2\pi}{T}$

(a) (*2 points*) Show that $[\hat{P}, \hat{Q}] = -i$.

(b) (*2 points*) Let us define

$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{Q} + i\hat{P}),$$
$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{Q} - i\hat{P}).$$

Calculate $\hat{a}^\dagger\hat{a}$ in terms of $\hat{P}$ and $\hat{Q}$.

(c) (*1 point*) Show that the Hamiltonian may be written as

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + 1/2\right).$$

(d) (*1 point*) Using the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$, show that the Hamiltonian may be written as

$$\hat{H} = \hbar\omega\left(\hat{a}\hat{a}^\dagger - 1/2\right).$$

2. A quantum system described by the Hamiltonian $\hat{H}$ is initially in the state

$$|\psi\rangle = N\left[\sqrt{2}|\phi_1\rangle + \sqrt{3}|\phi_2\rangle + |\phi_3\rangle + |\phi_4\rangle\right],$$

where $N \in \mathbb{R}$, $|\phi_n\rangle$ are the orthonormal eigenstates of energy such that $\hat{H}|\phi_n\rangle = nE_0|\phi_n\rangle$, and E_0 is a real-valued constant with units of energy.

(a) (*2 points*) Find a value for N such that $|\psi\rangle$ is normalized to unity.

(b) (*2 points*) Let the energy of $|\psi\rangle$ be measured. Give all possible measurement results and calculate their corresponding probabilities. Assume that the measurement is ideal, i.e., no measurement errors occur.

(c) (*2 points*) Consider an operator $\hat{X}$, the action of which on $|\phi_n\rangle$ $(n = 1, 2, 3, 4)$ is defined by $\hat{X}|\phi_n\rangle = (n+2)x_0|\phi_n\rangle$, where x_0 is a real-valued scalar. Suppose that a measurement of the energy of the above-defined $|\psi\rangle$ yields $4E_0$. Assume that immediately afterwards, we ideally measure the physical quantity corresponding to $\hat{X}$. What is the value for the quantity obtained in the latter measurement?

3. Consider a qubit described by the Hamiltonian $\hat{H} = \epsilon(-\mathrm{i}|0\rangle\langle 1| + \mathrm{i}|1\rangle\langle 0|)$, where $\{|0\rangle, |1\rangle\}$ form an orthonormal basis of the Hilbert space and ϵ is a real-valued constant with units of energy.

(a) (*1 point*) Show that the Hamiltonian is Hermitian.

(b) (*1 point*) Write the matrix representation of the Hamiltonian in the basis $\{|0\rangle, |1\rangle\}$.

(c) (*2 points*) Find the eigenenergies E_1 and E_2 of $\hat{H}$ and the corresponding eigenstates.

(d) (*2 points*) Suppose that initially (at $t = 0$) the system is in the state $|\psi(0)\rangle = |1\rangle$. Find the state $|\psi(t)\rangle$ at an arbitrary time t.

4. Briefly define the following terms:

(a) (*1 point*) Unitary quantum evolution

(b) (*1 point*) Complete basis for an arbitrary Hilbert space

(c) (*1 point*) Commutation relation of operators

(d) (*1 point*) Qubit

(e) (*2 points*) Heisenberg uncertainty principle

5. A point-like particle with mass m is moving freely inside a one-dimensional box with two infinite walls at $x = 0$ and $x = a$ and zero potential between the walls.

   (a) (*4 points*) Solve the wave functions of the eigenstates $\{\psi_n(x)\}_{n=1}^{\infty}$ and the eigenenergies $\{E_n\}_{n=1}^{\infty}$ of the particle.

   (b) (*1 point*) Let the energy of the ground state be $E_1 = 38$ eV. Find the energy of the particle in its first excited state E_2.

   (c) (*1 point*) Suppose that the particle is in the ground state. Let us suddenly double the size of the box, i.e., the right wall is moved instantaneously from $x = a$ to $x = 2a$. The quantum state of the particle does not change during the change. Find the probability that a subsequent measurement of the energy of the particle will yield the ground-state energy of the new box.

   Hint:

$$\int \sin(ax)\sin(bx)dx = \frac{\sin((a-b)x)}{2(a-b)} - \frac{\sin((a+b)x)}{2(a+b)}$$

Bonus: (*0.5 points*) How long did it take for you to finish the exam?