PHYS-C0252 — Quantum Mechanics

Final examination

Wednesday 07 December 2022, 13:00–16:00

Instructions

- Your answers should be legible and appropriately numbered.
- Your answers should contain relevant intermediate steps and explanations for the calculations.
- No calculators, cheat sheets or other materials are allowed.
- Return the exam paper when you are finished.

1. Briefly define the following terms:

   (a) (*1 point*) Quantum state of a physical system

   (b) (*1 point*) Eigenstate

   (c) (*1 point*) Hamiltonian operator of a physical system

   (d) (*1 point*) Bloch sphere

   (e) (*2 points*) The difference between fermions and bosons. Give at least two examples.

2. A quantum system described by a Hamiltonian $\hat{H}$ is initially in the state

$$|\psi\rangle = N\left[|\phi_1\rangle + |\phi_2\rangle + \frac{1}{\sqrt{2}}(1+\mathrm{i})\,|\phi_3\rangle + \sqrt{5}|\phi_4\rangle\right],$$

   where $|\phi_n\rangle$ are the eigenstates of energy such that $\hat{H}|\phi_n\rangle = nE_0|\phi_n\rangle$, E_0 has units of energy, and $N \in \mathbb{R}$.

   (a) (*1.5 points*) Find a suitable scalar N such that $|\psi\rangle$ is normalized.

   (b) (*1.5 points*) Let the energy of $|\psi\rangle$ be measured. Give all possible measurement results and their corresponding probabilities. Assume that the measurement is ideal, i.e., no measurement errors occur.

   (c) (*1.5 points*) What is the expectation value of the energy when the system is in the state $|\psi\rangle$? Is it a possible measurement result if the energy is measured?

(d) (*1.5 points*) Consider an operator $\hat{X}$, the action of which on $|\phi_n\rangle$ $(n = 1, 2, 3, 4)$ is defined by $\hat{X}|\phi_n\rangle = (n+2)x_0|\phi_n\rangle$, where x_0 is a real-valued scalar. Suppose that a measurement of the energy of the above-defined $|\psi\rangle$ yields $3E_0$. Assume that immediately afterwards, we ideally measure the physical quantity corresponding to $\hat{X}$. What is the value for the quantity obtained in the latter measurement?

3. Consider a complete orthonormal basis $\{|n\rangle\}_{n=0}^{\infty}$ for a Hilbert space $\mathcal{H}$ and define operators $\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle$ and $\hat{a}^{\dagger}\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$ such that $\hat{a}\,|0\rangle = 0$. Let us consider the Hamiltonian

$$\hat{H} = \hbar\omega\left(\hat{a}^{\dagger}\hat{a} + 1/2\right),$$

which in fact is the Hamiltonian of a quantum harmonic oscillator with eigenstates $\{|n\rangle\}_{n=0}^{\infty}$.

(a) (*1 point*) What are the eigenenergies ϵ_n of $\hat{H}$ in terms of $\hbar\omega$?

(b) (*1 point*) Show that $[\hat{a}, \hat{a}^{\dagger}] = 1$. (Hint: Operate on an arbitrary eigenstate $|n\rangle$.)

(c) (*1 point*) Show that $[\hat{N}, \hat{a}] = -\hat{a}$, where $\hat{N} = \hat{a}^{\dagger}\hat{a}$ is the number operator.

(d) (*1 point*) Show that $[\hat{N}, \hat{a}^{\dagger}] = \hat{a}^{\dagger}$.

(e) (*2 points*) Show that $|n\rangle = (n!)^{-1/2}(a^{\dagger})^n\,|0\rangle$.

4. Consider a system described by the Hamiltonian $\hat{H} = \epsilon(|0\rangle\langle 1| + |1\rangle\langle 0|)$, where $\{|0\rangle, |1\rangle\}$ form an orthonormal basis of the considered Hilbert space and ϵ is a real-valued constant with the dimension of energy.

(a) (*2 points*) Express the eigenenergies E_k and eigenvalues $|\psi_k\rangle$ of $\hat{H}$ in terms of ϵ, $|0\rangle$ and $|1\rangle$. Write the Hamiltonian in the form

$$\hat{H} = \sum_k E_k\,|\psi_k\rangle\langle\psi_k|\,.$$

(b) (*2 points*) Write the matrix representation $U(t)$ of the time evolution operator $\hat{U}(t)$ in the eigenbasis $\{|\psi_k\rangle\}$ using its matrix elements

$$U_{k,j}(t) := \langle\psi_k|\,\hat{U}(t)\,|\psi_j\rangle = \langle\psi_k|\,\mathrm{e}^{-\mathrm{i}\hat{H}t/\hbar}\,|\psi_j\rangle\,,$$

and write $\hat{U}(t)$ in the form

$$\hat{U}(t) = \sum_{k,j} U_{k,j}(t)\,|\psi_k\rangle\langle\psi_j|\,.$$

Express the result again in terms of ϵ, $|0\rangle$ and $|1\rangle$. Hint: $\mathrm{e}^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$.

(c) (*2 points*) Suppose that at $t = 0$, the system is in the state $|\psi(0)\rangle = |1\rangle$. Find the state $|\psi(t)\rangle$ at an arbitrary time t.

5. Consider a particle with wave function $\psi(x)$ and mass m in the step potential

$$V(x) = \begin{cases} 0, & x < 0, \\ V_0, & x > 0, \end{cases}$$

where $V_0 > 0$ has units of energy.

(a) (*2 points*) Write the stationary Schrödinger equation for a particle in this potential and write down the appropriate boundary conditions when the energy of the particle E satisfies $E > V_0$.

(b) (*2 points*) Show that in the opposite case, when $E < V_0$, the solutions to the stationary Schrödinger equation are of the form

$$\psi(x) = \begin{cases} A_I e^{ikx} + A_R e^{-ikx}, & x < 0, \\ B e^{-k_0 x}, & x > 0. \end{cases}$$

Express k and k_0 in terms of m, E and V_0.

(c) (*2 points*) Using the appropriate boundary conditions, calculate the reflection coefficient $R = |A_R/A_I|^2$ for waves incoming from $x < 0$, when $E < V_0$.

Bonus: (*0.5 points*) How long did it take for you to finish the exam?