Exercise 01 (70%). Consider a perfectly stirred biological reactor in which the biomass concentration $x_1(t)$ and the substrate concentration $x_2(t)$ are the state variables, and the substrate concentration in the feed $u_1(t)$ and the dilution rate $u_2(t)$ (the reciprocal of the residence time) are the control variables. The system evolves in time according to the dynamic model

$$\dot{x}_1(t) = \underbrace{x_1(t)\left[\mu(x_2(t)) - u_2(t)\right]}_{f_1(x(t),u(t)|\theta_x)}; \tag{1a}$$

$$\dot{x}_2(t) = \underbrace{u_2(t)\left[u_1(t) - x_2(t)\right] - \frac{1}{2}\mu(x_2(t))x_1(t)}_{f_1(x(t),u(t)|\theta_x)}, \tag{1b}$$

where $\mu(x_2(t)) = \dfrac{\theta_{x,1}}{\theta_{x,2} + x_2(t)} x_2(t)$ is the Monod growth rate with parameters θ_x taking the values $\theta_{x,1} = 0.6\text{h}^{-1}$ and $\theta_{x,2} = 0.2\text{g l}^{-1}$. We assume that the biomass concentration is measured: Thus, we also have the output equation

$$y(t) = \underbrace{x_1(t)}_{g(x(t),u(t)|\theta_y)} \tag{2}$$

For certain values $u^{\text{SS}} = (u_1^{\text{SS}}, u_2^{SS})$ of the controls, the bioreactor presents a stationary state $x^{\text{SS}} = (x_1^{\text{SS}}, x_2^{\text{SS}})$. Determine the linear state-space model

$$\dot{x}'(t) = Ax'(t) + Bu'(t), \tag{3a}$$

$$y'(t) = Cx'(t) + Du'(t); \tag{3b}$$

which approximates model (1) and (2) around the point $P = (u^{\text{SS}}, x^{\text{SS}})$.

Remembering that the perturbation variables are defined as $x'(t) = x(t) - x^{\text{SS}}$, $u'(t) = u(t) - u^{\text{SS}}$, and $y'(t) = y(t) - y^{\text{SS}}$, show the detailed steps needed to determine the matrices A, B, C, and D and the expressions obtained for each of their elements. Assume that only the dilution rate is known, $u_2^{\text{SS}} = 0.4\text{h}^{-1}$.

Exercise 02 (30%). Consider the following approximation of a process model

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 & 0 \\ 2 & -4 & 0 \\ 1 & 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t); \tag{4a}$$

$$y(t) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \end{bmatrix} u(t) \tag{4b}$$

(10%) Determine the stability of the system based on (the eigenvalues of) A;

(10%) Determine the controllability of the system based on the controllability matrix $\mathcal{C} = [B|AB|\cdots|A^{N_x-1}B]$, N_x is the dimensionality of the state;

(10%) Determine the system observability from the observability matrix $\mathcal{O}$.

This is an open-book examination. In addition to pencil/pen, eraser and other essential writing material, students are allowed to use own printed copies of the course material and own personal notes/scribbles.