ELEC-A7200 Signals and systems

Midterm 2

09.12.2022

Task 1

Pulse

$$x(t) = \begin{cases} t, & 0 < t \leq \frac{3}{2} \\ 3 - t, & \frac{3}{2} < t \leq 3 \\ 0, & \text{else,} \end{cases}$$

is sampled with sampling period $T_s = 1$ on interval $[0, 3]$.

a) (1p.) Find sampling sequence $\{x(n)\} = \{x(0), x(1), x(2), x(3)\}$.

b) (3p.) Find the Discrete Fourier Transform (DFT) $\{X(k)\} = \{X(0), X(1), X(2), X(3)\}$, när $X(k) = \sum_{n=0}^{2} x(n)e^{-j2\pi nk/3}$, $k = 0, \ldots, 2$.

c) (2p.) What frequencies do indexes $k = 0, 1, 2, 3$ in $\{X(k)\}$ correspond to?

d) (2p.) $\{x(n)\}$ is zero padded with eight zeros. How does this affect the frequency resolution of DFT?

e) (2p.) Explain why aliasing occurs when pulse $x(t)$ is sampled.

Task 2

Consider LTI-system whose input $x(t)$ and response $y(t)$ satisfy differential equation

$$2\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + y(t) = x(t).$$

a) (6p.) Find frequency response $H(f) = \frac{Y(f)}{X(f)}$ of the system.

Tip. $\mathcal{F}[\frac{d^ny(t)}{dt^n}] = (j2\pi f)^n Y(f)$

b) (4p.) Calculate system's amplitude response $A(f) = |H(f)|$ and phase response $\phi(f) = -\arg\{H(f)\}$ for frequencies 0.1 Hz and 10 Hz.

Task 3

Let the autocorrelation function of a random signal $\bar{\mathbf{y}}$ be

$$r_{\bar{y}\bar{y}}(\tau) = E\{\bar{y}(t)\bar{y}^*(t-\tau)\} = \exp(-|\tau|)$$

a) (2p.) Find average power $E\{|\bar{y}(t)|^2\}$ of $\bar{\mathbf{y}}$.

b) (3p.) Find power spectrum $S_{yy}(f)$ of the signal.

Let power spectrum of a white noise be $S_{zz}(f) = N_0/2$

c) (2p.) Bandwidth of some signal is B. Find noise power of the noise when it is bandlimited on the frequence band of that signal.

d) (3p.) Find frequency response $H(f)$ of a *stable* filter so that $S_{yy}(f) = |H(f)|^2 S_{zz}(f)$.

ELEC-A7200 Signaalit ja järjestelmät

Syksy 2022, 2. välikoe

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Tehtävä 1

Pulssia

$$x(t) = \begin{cases} t, & 0 < t \leq \frac{3}{2} \\ 3 - t, & \frac{3}{2} < t \leq 3 \\ 0, & \text{muulloin,} \end{cases}$$

näytteistetään välillä $[0, 3]$. Näyteväli $T_s = 1$.

a) (1p.) Ratkaise näytesekvenssi $\{x(n)\} = \{x(0), x(1), x(2), x(3)\}$.

b) (3p.) Ratkaise näytepisteiden $\{x(n)\}$ diskreetti Fourier-muunnos (DFT) $\{X(k)\} = \{X(0), X(1), X(2), X(3)\}$, missä $X(k) = \sum_{n=0}^{3} x(n) e^{-j2\pi nk/4}, k = 0, \ldots, 3$

c) (2p.) Mitä taajuuksia DFT:n indeksit $k = 0, 1, 2, 3$ vastaavat?

d) (2p.) Näytepisteisiin lisätään kahdeksan nollaa. Miten tämä vaikuttaa DFT:n taajuusresoluutioon?

e) (2p.) Selitä miksi pulssin $x(t)$ näytteenotossa tapahtuu aliasointia.

Tehtävä 2

LTI-järjestelmän heräte $x(t)$ ja vaste $y(t)$ toteuttavat differentiaaliyhtälön

$$2\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + y(t) = x(t).$$

a) (6p.) Ratkaise järjestelmän taajuusvaste $H(f) = \frac{Y(f)}{X(f)}$.

Vihje. $\mathcal{F}[\frac{d^n y(t)}{dt^n}] = (j2\pi f)^n Y(f)$

b) (4p.) Laske amplitudivaste $A(f) = |H(f)|$ ja taajuusvaste $\phi(f) = -\arg\{H(f)\}$ taajuuden arvoille 0.1 Hz ja 10 Hz.

Tehtävä 3

Erään satunnaissignaalin autokorrelaatiofunktio on

$$r_{\bar{y}\bar{y}}(\tau) = E\{\bar{y}(t)\bar{y}^*(t-\tau)\} = \exp(-|\tau|)$$

a) (2p.) Ratkaise signaalin keskimääräinen teho $E\{|\bar{y}(t)|^2\}$.

b) (3p.) Ratkaise signaalin tehospektri $S_{yy}(f)$.

Valkoisen kohinan tehospektri on $S_{zz}(f) = N_0/2$.

c) (2p.) Erään signaalin kaistanleveys on B. Ratkaise kohinan teho signaalin kaistalla.

d) (3p.) Ratkaise *stabiilin* suodattimen taajuusvaste $H(f)$ siten että $S_{yy}(f) = |H(f)|^2 S_{zz}(f)$.

Theorems of the fourier transform	Function	Transform
Linearity	$ax(t) + by(t)$	$aX(f) + bY(f)$
Time delay or time shift	$x(t-a)$	$X(f)e^{-j2\pi fa}$
Scale change	$x(at)$	$\frac{1}{\lvert a\rvert}X(\frac{f}{a})$
Conjugation	$x^*(t)$	$X^*(-f)$
Duality	$X(t)$	$x(-f)$
Frequency shift	$x(t)e^{j2\pi at}$	$X(f-a)$
Linear modulation	$x(t)\cos(2\pi at + b)$	$\frac{e^{jb}X(f-a)+e^{-jb}X(f+a)}{2}$
Differentiation	$\frac{d^n x(t)}{dt^n}$	$(j2\pi f)^n X(f)$
Integration	$\int_{-\infty}^{t} x(u)du$	$\frac{X(f)}{j2\pi f}$
Convolution	$x(t) \otimes y(t)$	$X(f)Y(f)$
Multiplication	$x(t)y(t)$	$X(f) \otimes Y(f)$
Multiplication by t^n	$t^n x(t)$	$-\frac{1}{j2\pi}\frac{d^n X(f)}{df^n}$

Fourier transforms	Function	Transform
Rectangular pulse	$\mathrm{rect}(t/a)$	$a \cdot \mathrm{sinc}(af)$
Triangular pulse	$\mathrm{tria}(t/a)$	$a \cdot \mathrm{sinc}^2(af)$
Gaussian pulse	$e^{-\pi(\frac{t}{a})^2}$	$a \cdot e^{-\pi(af)^2}$
One sided exponential pulse	$e^{-t/a}u(t)$	$\frac{a}{1+j2\pi fa}$
Two sided exponential pulse	$e^{-\lvert t\rvert/a}$	$\frac{2a}{1+(2\pi fa)^2}$
Sinc pulse	$\mathrm{sinc}(at)$	$\frac{1}{a}\mathrm{rect}(f/a)$
Constant	a	$a \cdot \delta(f)$
Phasor	$e^{j(2\pi at+b)}$	$e^{jb}\delta(f-a)$
Cosine wave	$\cos(2\pi at + b)$	$\frac{e^{jb}\delta(f-a)+e^{-jb}\delta(f+a)}{2}$
Delayed impulse	$\delta(t-a)$	$e^{-j2\pi fa}$
Step	$u(t)$	$\frac{\delta(f)}{2} + \frac{1}{j2\pi f}$

$$T_0 = \frac{1}{f_0} = \frac{2\pi}{\omega_0}$$

$$e^{j\phi} = \cos(\phi) + j\sin(\phi)$$

$$\sin^2\phi + \cos^2\phi = 1$$

$$\cos(\phi) = \sin(\phi + \pi/2)$$

$$\sin(\phi) = \cos(\phi - \pi/2)$$

$$\cos^2(\phi) = \frac{1}{2}\Big[1 + \cos(2\phi)\Big]$$

$$\sin^2(\phi) = \frac{1}{2}\Big[1 - \sin(2\phi)\Big]$$

$$\cos^3(\phi) = \frac{1}{4}\Big[3\cos(\phi) + \cos(3\phi)\Big]$$

$$\sin^3(\phi) = \frac{1}{4}\Big[3\sin(\phi) - \sin(3\phi)\Big]$$

$$\sin(\alpha)\cos(\beta) = \frac{\sin(\alpha - \beta) + \sin(\alpha + \beta)}{2}$$

$$\sin(\alpha)\sin(\beta) = \frac{\cos(\alpha - \beta) - \cos(\alpha + \beta)}{2}$$

$$\cos(\alpha)\cos(\beta) = \frac{\cos(\alpha - \beta) + \cos(\alpha + \beta)}{2}$$

$$x(t) = \sum_{k=-\infty}^{\infty} x_k e^{j2\pi k f_0 t} = \frac{\alpha_0}{2} + \sum_{k=1}^{\infty}\Big[\alpha_k \cos(2\pi k f_0 t) + \beta_k \sin(2\pi k f_0 t)\Big]$$

$$x_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-j2\pi k f_0 t}\, dt$$

$$\alpha_k = 2 \cdot \mathrm{Re}\{x_k\}, \quad \text{when } x(t) \in \mathbb{R}$$

$$\beta_k = -2 \cdot \mathrm{Im}\{x_k\}, \quad \text{when } x(t) \in \mathbb{R}$$

$$X(f) = \mathcal{F}\Big\{x(t)\Big\} = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t}\, dt$$

$$x(t) = \mathcal{F}^{-1}\Big\{X(f)\Big\} = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t}\, df$$

$$X_k = \sum_{n=0}^{N-1} x_n e^{-j2\pi kn/N}$$

$$x_k = \frac{1}{N}\sum_{n=0}^{N-1} X_n e^{j2\pi kn/N}$$

$$f_0 = \frac{1}{N \cdot T_s} = \frac{f_s}{N}$$

$$s = \sigma + j\omega = \sigma + j2\pi f$$

$$X(s) = \mathcal{L}\Big\{x(t)\Big\} = \int_0^{\infty} x(t) \cdot e^{-st}\, dt$$

$$d_n = \frac{u_n}{u_1}$$

$$d_{\mathrm{tot}} = \sqrt{\sum_{n=2}^{\infty} d_n^2}$$