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Department of Mathematics and Systems Analysis Korte

MS-C1350 Partial differential equations, fall 2022

Final exam (100%) on 12 Dec 2022 at 9:00-12:00

No calculators or other equipment except for pen and paper.

Remember to explain carefully your answers. Each of the problems 1–5 is worth 6 points. Answer all problems.

1. (a) Formulate in exact form the Dirichlet boundary value problem for the Laplace equation in the upper half-space $\mathbb{R}^{n+1}_+ = \{(x,y) : x \in \mathbb{R}^n,\ y > 0\}$.
   (b) Solve this problem using Fourier transform.
   (c) Is the solution to this problem unique? Motivate your answer.

2. Let $c > 0$ be a real constant and $u = u(x,t)$ a solution to the general wave equation $u_{tt} - c^2\Delta u = 0$.

   (a) For which values of $a, b \in \mathbb{R}$, the function $v(x,t) = u(ax, bt)$ is a solution to the equation $v_{tt} - \Delta v = 0$?
   (b) Give a formula for the solution to the problem

$$\begin{cases} u_{tt} - c^2\Delta u = 0 & \text{in} \quad \mathbb{R}^3 \times (0,\infty), \\ u = g \quad \text{and} \quad u_t = h & \text{in} \quad \mathbb{R}^3 \times \{t = 0\}. \end{cases}$$

   Hint: Kirchhoff's formula for the solutions to the problem $v_{tt} - \Delta v = 0$ in dimension 3 is

$$v(x,t) = \frac{1}{|\partial B(x,t)|}\int_{\partial B(x,t)} (th(y) + g(y) + \nabla g(y)\cdot(y-x))\,dS(y).$$

3. Find a solution to the following one-dimensional non-homogeneous problem for the wave equation:

$$\begin{cases} u_{tt} - u_{xx} = te^x, & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = 0, & x \in \mathbb{R}, \\ u_t(x,0) = 0, & x \in \mathbb{R}. \end{cases}$$

   Hints:
   Duhamel's principle for solving the non-homogeneous problem $u_{tt} - \Delta u = f$:

$$u(x,t) = \int_0^t u(x,t;s)\,ds, \quad x \in \mathbb{R}^n, \quad t > 0,$$

   where $u(x,t;s)$ is a solution to a homogeneous wave problem with initial time at $t = s$.
   d'Alembert's formula for the solution to the homogeneous problem in dimension one:

$$u(x,t) = \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy.$$

4. Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set.

   (a) What is the weak maximum principle for the Laplace equation in Ω?

   (b) What is the strong maximum principle for the Laplace equation in Ω?

   (c) How can you prove the uniqueness of the solution to the Dirichlet boundary value problem using maximum principle?

   (d) Is there a maximum principle for the heat equation and/or the wave equation? If yes, formulate **one** maximum principle for the heat equation **or** for the wave equation.

5. Let $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^n$ be a unit square. Assume that $g, h \in C(\partial\Omega)$. Consider the following mixed boundary value problem

$$\begin{cases} \Delta u = 0 \quad \text{in} \quad \Omega, \\ u = g \quad \text{on} \quad \partial\Omega \cap \{x = 0 \text{ or } x = 1\}, \\ \dfrac{\partial u}{\partial \nu} = h \quad \text{on} \quad \partial\Omega \cap \{y = 0 \text{ or } y = 1\}. \end{cases}$$

   Show that, if the problem above has a solution $u \in C^2(\overline{\Omega})$, then the solution is unique up to an additive constant.

   Hint: Green's first identity is

$$\int_\Omega \nabla u \cdot \nabla v \, dx = -\int_\Omega u \Delta v \, dx + \int_{\partial\Omega} \frac{\partial v}{\partial \nu} u \, dS.$$