Aalto University School of Science
Department of Mathematics and Systems Analysis Korte

MS-C1350 Partial differential equations, fall 2022

Course exam on 12 Dec 2022 at 9:00-12:00

No calculators or other equipment except for pen and paper.

This set of problems is for the participants of the course in the fall 2022 and affects 50% in the grading of the course.

For problems 2–5, remember to explain carefully your answers. Each of the problems 1–5 is worth 6 points. Answer all problems.

1. Which of the following claims are true? Copy the following table in your answer sheet and write **T** for true or **F** for false. You do not need to motivate your answers in this question.

A	B	C	D	E	F	G	H	I	J
T/F	...								

(Grading: N correct answers $\Longrightarrow$ $\max(N-4, 0)$ points.)

(A) The fundamental solution of the heat equation is a bounded function in $\mathbb{R}^{n+1}$.

(B) The fundamental solution of the heat equation converges to zero as $|x| \to 0$ or $t \to \infty$.

(C) The solution of the Cauchy problem for the heat equation depends only on the initial values near the origin.

(D) If $u = u(x)$ is a solution to the Laplace equation in $\mathbb{R}^n$, then $v(x,t) = u(x)$ is a solution to the heat equation in $\mathbb{R}^n \times (0, \infty)$.

(E) If $u = u(x,t)$ is a solution to the heat equation in $\mathbb{R}^n \times (0, \infty)$, then $v(x) = u(x,t)$ is a solution to the Laplace equation in $\mathbb{R}^n$ for every $t \in (0, \infty)$.

(F) If $u = u(x,t)$ is a solution to the heat equation in $\mathbb{R}^n \times (0, \infty)$, then $v(x,t) = u(x,-t)$ is a solution to the heat equation in $\mathbb{R}^n \times (-\infty, 0)$.

From now on, let us consider a solution of the heat equation in a bounded space-time cylinder Ω_T:

(G) The boundary values can be given on the whole boundary $\partial\Omega_T$.

(H) The solution at $t = 0$ is determined by the equation.

(I) The solution at $t = T$ is determined by the equation.

(J) If the initial values on $\Omega \times \{0\}$ are discontinuous, then the solution is discontinuous in Ω_T.

2. (a) Formulate in exact form the Dirichlet boundary value problem for the Laplace equation in the upper half-space $\mathbb{R}^{n+1}_+ = \{(x,y) : x \in \mathbb{R}^n,\ y > 0\}$.

   (b) Explain how this problem can be solved using the Fourier transform. It is enough to describe the main steps verbally.

   (c) Is the solution to this problem unique? Motivate your answer.

3. Let $c > 0$ be a real constant and $u = u(x,t)$ a solution to the general wave equation $u_{tt} - c^2 \Delta u = 0$.

   (a) Show that the function $v(x,t) = u(x, \frac{t}{c})$ is a solution to the equation $v_{tt} - \Delta v = 0$.

   (b) Give a formula for the solution to the problem

$$\begin{cases} u_{tt} - c^2 \Delta u = 0 \quad \text{in} \quad \mathbb{R}^3 \times (0, \infty), \\ u = g \quad \text{and} \quad u_t = h \quad \text{in} \quad \mathbb{R}^3 \times \{t = 0\}. \end{cases}$$

Hint: Kirchhoff's formula for the solutions to the problem $v_{tt} - \Delta v = 0$ in dimension 3 is

$$v(x,t) = \frac{1}{|\partial B(x,t)|} \int_{\partial B(x,t)} (th(y) + g(y) + \nabla g(y) \cdot (y - x))\, dS(y).$$

4. Find a solution to the following one-dimensional non-homogeneous problem for the wave equation:

$$\begin{cases} u_{tt} - u_{xx} = te^x, & \text{in } \mathbb{R} \times (0, \infty), \\ u(x,0) = 0, & x \in \mathbb{R}, \\ u_t(x,0) = 0, & x \in \mathbb{R}. \end{cases}$$

Hints:
Duhamel's principle for solving the non-homogeneous problem $u_{tt} - \Delta u = f$:

$$u(x,t) = \int_0^t u(x,t;s)\, ds, \quad x \in \mathbb{R}^n, \quad t > 0,$$

where $u(x,t;s)$ is a solution to a homogeneous wave problem with initial time at $t = s$.

d'Alembert's formula for the solution to the homogeneous problem in dimension one:

$$u(x,t) = \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2} \int_{x-t}^{x+t} h(y)\, dy.$$

5. Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set.

   (a) What is the weak maximum principle for the Laplace equation in Ω?

   (b) What is the strong maximum principle for the Laplace equation in Ω?

   (c) How can you prove the uniqueness of the solution to the Dirichlet boundary value problem using maximum principle?

   (d) Is there a maximum principle for the heat equation and/or the wave equation? If yes, formulate **one** maximum principle for the heat equation **or** for the wave equation.