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Exam, Monday, December 5, 2022, 09:00 - 12:00
Complex Analysis, MS-C1300

Motivate your answers. Only giving answers gives no points. No calculators or books are allowed. **Good luck!**

(1) (a) Find the Taylor series of

$$f(z) = \frac{1}{(1-z)^2}$$

around $z_0 = 0$. Determine the radius of convergence ρ for the series. (2p)

(b) Find the Taylor series of

$$f(z) = \frac{1}{2-z-z^2}$$

around $z_0 = 0$. Determine the radius of convergence ρ for the series. (*Hint:* Partial fractions help.) (2p)

(c) Find the Laurent series of

$$f(z) = \frac{\operatorname{Log} z}{(z-1)^2}$$

in $\{z \in \mathbb{C}; 0 < |z-1| < 1\}$. (2p)

(2) Calculate

$$\int_\gamma \frac{\operatorname{Log}(1+z)}{(2z-1)^3} dz$$

where:

(a) $\gamma(t) = \frac{e^{it}}{2} + \frac{1}{2}$, for $0 \le t \le 2\pi$. (3p)

(b) $\gamma(t) = (2\cos t - 1)e^{it}$, for $0 \le t \le 2\pi$. (3p)

(*Hint:* Be careful with the winding number in (b). Try to sketch the curve. Think about for which t, $2\cos t - 1$ is positive, when it is zero, and when negative.)

(3) Let

$$f(z) = \frac{z^2-1}{(z^2+1)^2}.$$

(a) Find the poles of f and determine their order. Calculate the residues of f at those poles. (3p)

(b) Calculate

$$\int_{-\infty}^{\infty} \frac{x^2-1}{(x^2+1)^2}\,dx$$

(3p)

(4) Let U be a domain in $\mathbb{C}$ and assume that $f = u + iv\colon U \to \mathbb{C}$ is an analytic function. Also assume that $g = u^2 + iv^2\colon U \to \mathbb{C}$ is analytic. Prove that f is a constant function. (*Hint:* Use the Cauchy–Riemann equations.) (6p)

Useful formulas

- **Cauchy–Riemann equations**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

- **Cauchy's Integral Formula**

$$\eta(\gamma, z_0) f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

- **Residue for a pole of order m**

$$\mathrm{Res}(z_0, f) = \frac{1}{(m-1)!}\lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]$$

- **Some Taylor series**

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \text{ when } |z| < 1$$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \text{ when } z \in \mathbb{C}$$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \text{ when } z \in \mathbb{C}$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \text{ when } z \in \mathbb{C}$$