MS-C2111 Stochastic processes
Department of Mathematics and Systems Analysis
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Exam
7.12.2022, 16:30–19:30
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The exam consists of 4 problems, each worth 6 points. You are **not** allowed to use a calculator in the exam. However, you are allowed to use a **handwritten memory aid sheet** of size A4, with text only on one side and with your name and student number in the upper right corner.

1 Consider processes on state space $S = \{-1, 0, 1\}$, and consider the following matrices:

$$M_1 = \begin{bmatrix} 0.2 & 0.5 & 0.2 \\ 0.8 & 0 & 0.2 \\ 0 & 0.5 & 0.6 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0.2 & 0.8 & 0 \\ 0.5 & 0 & 0.5 \\ 0.2 & 0.2 & 0.6 \end{bmatrix},$$

$$M_4 = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -3 & 2 \\ 0 & 2 & -2 \end{bmatrix}, \quad M_5 = \begin{bmatrix} -1 & 0 & 2 \\ 2 & -3 & 2 \\ 0 & 2 & -1 \end{bmatrix}$$

Justify your answers to the following questions briefly.

(a) **(1p.)** Which of the matrices are transition matrices?

(b) **(1p.)** Which of the transition matrices have unique stationary distribution?

(c) **(1p.)** Take one of the transition matrices. Let $X = (X_t)_{t \in \mathbb{N}}$ be the corresponding Markov chain. Given that $X_5 = -1$, what is the probability of $X_7 = -1$?

(d) **(1p.)** Which of the matrices are generator matrices?

(e) **(1p.)** Take one of the generator matrices. Let $Z = (Z_t)_{t \in \mathbb{R}_+}$ be the corresponding continuous time Markov chain. Compute the expected exit time of Z from $Z_0 = 0$.

(f) **(1p.)** Given that $Z_0 = 0$, what is the probability that the first jump of the process goes to the state 1?

2 During a meteor shower, amateur astronomer Aurora observes shooting stars. According to the forecast, the meteor shower lasts 2 hours.

The shooting stars seem to arrive according to a Poisson process $N = (N(t))_{t \in \mathbb{R}_+}$ with intensity $\lambda = 1/2$ (in units $\frac{1}{\text{min}}$, where the unit of time t is minutes, min).

Justify your answers to the following questions briefly. [You don't have to numerically evaluate powers of e.]

(a) **(1p.)** What is the probability $\mathbb{P}[N(4) = 4]$ that during four minutes Aurora observes four shooting stars?

(b) **(2p.)** What is the conditional probability $\mathbb{P}[N(30) = 20 \mid N(22) = 16]$ that during half an hour, Aurora observes 20 shooting stars in total, given that 16 of them are observed during the first 22 minutes?

(c) **(1p.)** Which are the most likely numbers of shooting stars observed by Aurora during the first four minutes?

Disappointingly, it soon turns out that Aurora has mistakenly been regarding also aeroplanes as shooting stars. In fact, 2/3 of the observations are actually aeroplanes.

(d) **(2p.)** Knowing this new information, what is the expected number of shooting stars observed by Aurora during the meteor shower?

3 Olive's garden has been infested by flower eating pests. Their lifespan is one day, and they lay eggs in the evening independently of each other. The number of eggs laid by each individual, denoted by Y, follows the geometric distribution:

$$\mathbb{P}[Y = k] = (1-p)p^k, \qquad k \in \{0, 1, 2, \dots\}.$$

To combat pests, every night Olive sprays λ decilitres of pesticides to her garden. As a result, in the morning each egg hatches with probability $e^{-\alpha\lambda}$ independently from each other. Unhatched eggs won't be able to hatch in the future.

(a) **(3p.)** Denote by Y_λ the number of surviving offspring (that is, the number of hatched eggs) from an individual pest. Here, you will compute its generating function $\phi_\lambda(x) = \mathbb{E}[x^{Y_\lambda}]$. [If you get stuck, you can still proceed to parts (b) & (c).]

(i) Show that

$$\mathbb{P}[Y_\lambda = n] = (1-p)\sum_{m=0}^{\infty} \binom{n+m}{n} (pe^{-\alpha\lambda})^n (p(1-e^{-\alpha\lambda}))^m.$$

Hint: Consider the events $\{Y_\lambda = n\}$, and $\{Y = m+n\}$ for $n, m \in \mathbb{N}$ separately.

(ii) Show that

$$\phi_\lambda(x) = \left(1 - \frac{pe^{-\alpha\lambda}(x-1)}{1-p}\right)^{-1}.$$

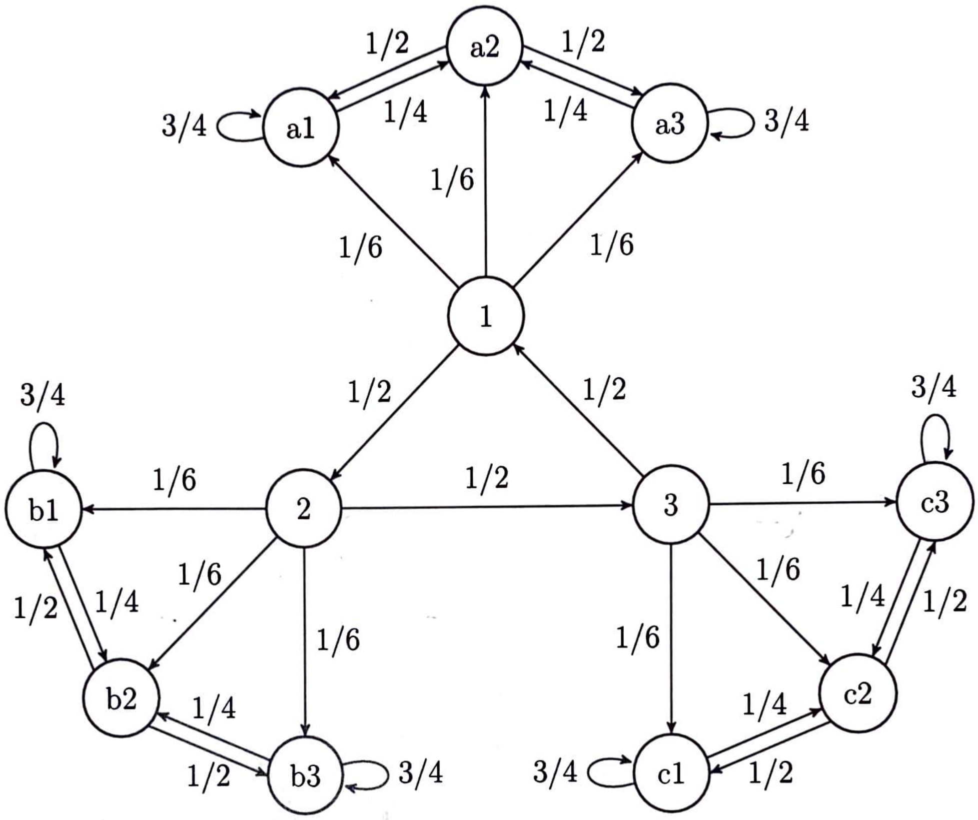
Hint: You can use the following change of summation indices:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f(n, n+m) = \sum_{m=0}^{\infty}\sum_{n=0}^{m} f(n, m).$$

(b) **(2p.)** Suppose that at the time Olive starts to use pesticides, there are a total of $X_0 = n$ pests in her garden. Denote by X_t the number of pests in the garden on day $t \in \{1, 2, \dots\}$. Let $T_0 = \min\{t \geq 0 : X_t = 0\}$ be the extinction time of the pests. Compute the extinction probability $\mathbb{P}[T_0 < \infty]$.

(c) **(1p.)** Compute $\mathbb{E}[Y_\lambda]$. **Hint:** You can use the generating function.

4 Consider a Markov chain $X = (X_t)_{t\in\mathbb{N}}$ described by the following transition diagram:



Write $A = \{\mathrm{a1}, \mathrm{a2}, \mathrm{a3}\}$, $B = \{\mathrm{b1}, \mathrm{b2}, \mathrm{b3}\}$, and $C = \{\mathrm{c1}, \mathrm{c2}, \mathrm{c3}\}$. Let τ_A, τ_B, and τ_C be the times when X hits the sets A, B, and C, respectively. For example, τ_A is defined as

$$\tau_A = \min\{t \geq 0 : X_t \in A\}.$$

Justify your answers to the following questions briefly.

(a) **(2p.)** Given that $X_0 = 1$, compute the hitting probabilities

$$\mathbb{P}[\tau_A < \infty], \qquad \mathbb{P}[\tau_B < \infty], \qquad \text{and} \qquad \mathbb{P}[\tau_C < \infty].$$

(b) **(2p.)** Given that $X_0 = 1$, what are the invariant distributions of X?

(c) **(2p.)** Given that $X_0 = 1$, what is the limiting distribution of X, if any?