PHYS-E0461 Introduction to plasma physics for fusion and space applications
Second examination 12.12.2022

Problems from previous exams:

1. (**Understanding plasmas**) (6p)

   (a) Explain briefly how collisions differ in neutral gas, weakly ionized gas, and fully ionized plasma. (2p)

   (b) Explain the major differences (advantages and disadvantages) between a tokamak and a stellarator as far as fusion energy production is concerned. (2p)

   (c) Explain (you do NOT need to remember the names) the structure of Earth's atmosphere, i.e., on what basis the atmosphere is divided into different spheres. Also tell for which layers the name 'sphere' is not appropriate. (2p)

2. (**Conductivities**) (6p)

   Using a simple steady-state fluid equation for the electrons, $0 = -e(\boldsymbol{E} + \boldsymbol{v}_e \times \boldsymbol{B}) - \nu_c m_e \boldsymbol{v}_e$, where ν_c is the collision frequency, and assuming stationary ions, show that Ohm's law can be expressed as a matrix equation $\boldsymbol{J} = \sigma \cdot \boldsymbol{E}$, where

$$\sigma = \begin{pmatrix} \sigma_P & -\sigma_H & 0 \\ \sigma_H & \sigma_P & 0 \\ 0 & 0 & \sigma_{\parallel} \end{pmatrix}$$

   when the z-axis is taken along the constant magnetic field, $\boldsymbol{B} = B\hat{z}$. Here, $\sigma_P = \frac{\nu_c^2}{\nu_c^2+\Omega^2}\sigma_{\parallel}$ is the so-called Pedersen conductivity, $\sigma_H = \frac{\nu_c \Omega}{\nu_c^2+\Omega^2}\sigma_{\parallel}$ is the so-called Hall conductivity, $\sigma_{\parallel} = \frac{ne^2}{m_e \nu_c}$ is the parallel conductivity, and Ω the electron cyclotron frequency. What can you say about the plasma conductivity along the magnetic field compared to its value across the magnetic field?

3. (**Plasma stability in a cylinder**) (6p)

   Consider a cylindrically symmetric plasma column ($\partial_z = 0, \partial_\theta = 0$; z is the direction of the cylinder axis) under equilibrium conditions, confined by a magnetic field. Verify that in cylindrical coordinates the hydromagnetic equilibrium is given by

$$\frac{dp\,(r)}{dr} = j_\theta\,(r)\, B_z\,(r) - j_z\,(r)\, B_\theta\,(r)\,. \text{ (2p)}$$

   Using Ampere's law, derive the equilibrium equation to the form

$$\frac{d}{dr}\left(p + \frac{B_\theta^2}{2\mu_0} + \frac{B_z^2}{2\mu_0}\right) = -\frac{B_\theta^2}{\mu_0 r}. \text{ (3p)}$$

   Give a physical interprepation why the two terms including magnetic field are organized to the left hand side and one to the right hand side. (1p)

4. (**Magnetic dipole field**) (6p)

   Recall from basic electromagnetism: far enough from the magnetic dipole with magnetization M, the field can be expressed as $\boldsymbol{B} = -\nabla\Psi$, where $\Psi = -(\mu_0/(4\pi))\boldsymbol{M} \cdot \nabla(1/r)$.

   (a) Using spherical coordinates and setting the z-axis along the magnetic dipole, show that the magnetic field is

$$\boldsymbol{B} = \frac{\mu_0}{4\pi}\frac{(3\boldsymbol{M} \cdot \hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{M}}{r^3}.$$

   (2p)

   (b) Calculate the components of the magnetic field (still in spherical coordinates) as well as the magnitude of the magnetic field. (2p)

   (c) In space and geophysics, instead of the conventional azimuthal angle θ, ranging from 0 (north pole) to π (south pole), one typically uses another angle λ, which measures the azimuthal angle not from the z-axis but, rather, from the (x,y) plane. So, λ ranges from $-\pi/2$ (north pole) to $+\pi/2$ (south pole). Express the magnetic field components in terms of λ. (2p)

1 Helpful physics formulas

Debye length	$\lambda_D^2 = \frac{\epsilon_0 T_e}{e^2 n_e}$	(1)
Plasma parameter	$\Lambda = \frac{4}{3} n\pi\lambda_D^3$	(2)
Plasma frequency	$\omega_p^2 = \frac{e^2 n}{m\epsilon_0}$	(3)
Larmor frequency	$\Omega = \frac{qB}{m}$	(4)
Larmor radius	$r_L = \frac{mv_\perp}{qB}$	(5)
Magnetic moment	$\mu = \frac{mv_\perp^2}{2B}$	(6)
$\boldsymbol{E} \times \boldsymbol{B}$ drift	$\boldsymbol{v}_{\boldsymbol{E}\times\boldsymbol{B}} = \frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2}$	(7)
Gradient drift	$\boldsymbol{v}_{\nabla B} = \pm\frac{1}{2} v_\perp r_L \frac{\boldsymbol{B}\times\nabla B}{B^2}$	(8)
Diamagnetic drift	$\boldsymbol{v}_D = -\frac{\nabla p \times \boldsymbol{B}}{qnB^2}$	(9)
Maxwell-Boltzmann	$f(\boldsymbol{v}) = \left(\frac{m}{2\pi T}\right)^{3/2} \exp\left(-\frac{mv^2}{2T}\right)$	(10)
	$h(E) = \frac{2}{T^{3/2}}\sqrt{\frac{E}{\pi}} e^{-E/T}$	(11)
Convective derivative	$\frac{\mathrm{d}n(\boldsymbol{r},t)}{\mathrm{d}t} = \frac{\partial n(\boldsymbol{r},t)}{\partial t} + \boldsymbol{v}\cdot\nabla n(\boldsymbol{r},t)$	(12)
Collision frequency	$\nu = \frac{e^4 \ln\Lambda}{4\pi\epsilon_0^2\sqrt{m}} \frac{n}{T^{3/2}}$	(13)
Gauss's law	$\nabla\cdot\boldsymbol{E} = \frac{\rho}{\epsilon_0}$	(14)
Gauss's law for magnetism	$\nabla\cdot\boldsymbol{B} = 0$	(15)
Maxwell-Faraday equation	$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}$	(16)
Ampère's circuital law	$\nabla\times\boldsymbol{B} = \mu_0\boldsymbol{J} + \frac{1}{c^2}\frac{\partial\boldsymbol{E}}{\partial t}$	(17)
Sound speed in ideal gases	$v_s = \sqrt{\frac{E_{\text{therm}}}{m}}$	(18)
Sound speed in plasma	$v_s = \sqrt{\frac{E_{\text{therm, e}}}{M} + \frac{\gamma_i E_{\text{therm, i}}}{M}}$	(19)

2 Constants

- vacuum permittivity $\epsilon_0 = 8.9 \cdot 10^{-12}$ C/Vm
- magnetic permeability $\mu_0 = 4\pi \times 10^{-7}$ Tm/A
- speed of light $c = 3 \times 10^8$ m/s
- elementary charge $e = 1.6 \cdot 10^{-19}$ C
- electron mass $m_e = 9.1 \cdot 10^{-31}$ kg
- proton mass $m_p = 1.7 \times 10^{-27}$ kg

3 Vector identities

(a) $\boldsymbol{A}\cdot\boldsymbol{B}\times\boldsymbol{C}=\boldsymbol{A}\times\boldsymbol{B}\cdot\boldsymbol{C}=\boldsymbol{B}\cdot\boldsymbol{C}\times\boldsymbol{A}$

(b) $\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})=(\boldsymbol{C}\times\boldsymbol{B})\times\boldsymbol{A}=\boldsymbol{B}(\boldsymbol{A}\cdot\boldsymbol{C})-\boldsymbol{C}(\boldsymbol{A}\cdot\boldsymbol{B})$

(c) $(\boldsymbol{A}\times\boldsymbol{B})\cdot(\boldsymbol{C}\times\boldsymbol{D})=(\boldsymbol{A}\cdot\boldsymbol{C})(\boldsymbol{B}\cdot\boldsymbol{D})-(\boldsymbol{A}\cdot\boldsymbol{D})(\boldsymbol{B}\cdot\boldsymbol{C})$

(d) $(\boldsymbol{A}\times\boldsymbol{B})\times(\boldsymbol{C}\times\boldsymbol{D})=\boldsymbol{C}(\boldsymbol{A}\times\boldsymbol{B}\cdot\boldsymbol{D})-\boldsymbol{D}(\boldsymbol{A}\times\boldsymbol{B}\cdot\boldsymbol{C})$

(e) $\nabla\cdot(f\boldsymbol{A})=f(\nabla\cdot\boldsymbol{A})+\boldsymbol{A}\cdot(\nabla f)$

(f) $\nabla\times(f\boldsymbol{A})=f(\nabla\times\boldsymbol{A})-\boldsymbol{A}\times(\nabla f)$

(g) $\nabla\cdot(\boldsymbol{A}\times\boldsymbol{B})=\boldsymbol{B}\cdot(\nabla\times\boldsymbol{A})-\boldsymbol{A}\cdot(\nabla\times\boldsymbol{B})$

(h) $\nabla\times(\boldsymbol{A}\times\boldsymbol{B})=(\boldsymbol{B}\cdot\nabla)\boldsymbol{A}-(\boldsymbol{A}\cdot\nabla)\boldsymbol{B}+\boldsymbol{A}(\nabla\cdot\boldsymbol{B})-\boldsymbol{B}(\nabla\cdot\boldsymbol{A})$

(i) $\nabla(\boldsymbol{A}\cdot\boldsymbol{B})=\boldsymbol{A}\times(\nabla\times\boldsymbol{B})+\boldsymbol{B}\times(\nabla\times\boldsymbol{A})+(\boldsymbol{A}\cdot\nabla)\boldsymbol{B}+(\boldsymbol{B}\cdot\nabla)\boldsymbol{A}$

(j) $\nabla\cdot(\boldsymbol{A}\boldsymbol{B})=(\nabla\cdot\boldsymbol{A})\boldsymbol{B}+(\boldsymbol{A}\cdot\nabla)\boldsymbol{B}$

(k) $\nabla\cdot(f\mathbf{T})=\nabla f\cdot\mathbf{T}+f\nabla\cdot\mathbf{T}$

(l) $\nabla\cdot(\nabla\times\boldsymbol{A})=0$

(m) $\nabla^2\boldsymbol{A}=\nabla(\nabla\cdot\boldsymbol{A})-\nabla\times\nabla\times\boldsymbol{A}$

(n) $\nabla(fg)=f\nabla g+g\nabla f$

(o) $\nabla\cdot(\nabla f\times\nabla g)=0$

(p) $\nabla\cdot\nabla f=\nabla^2 f$

(q) $\nabla\times\nabla f=0$

(r) $\int_V \nabla f\,\mathrm{d}V=\int_S f\,\mathrm{d}\boldsymbol{S}$

(s) $\int_V \nabla\times\boldsymbol{A}\,\mathrm{d}V=\oint_S \mathrm{d}\boldsymbol{S}\times\boldsymbol{A}$

(t) $\int_S \mathrm{d}\boldsymbol{S}\cdot\nabla\times\boldsymbol{A}=\oint_C \mathrm{d}\boldsymbol{l}\cdot\boldsymbol{A}$

(u) $\oint_C \mathrm{d}\boldsymbol{l}\times\boldsymbol{A}=\int_S(\mathrm{d}\boldsymbol{S}\times\nabla)\times\boldsymbol{A}$

4 Curvilinear coordinates

Cylindrical Coordinates (r,φ,z)

(a) $\nabla f=\frac{\partial f}{\partial r}\hat{\boldsymbol{r}}+\frac{1}{r}\frac{\partial f}{\partial\varphi}\hat{\boldsymbol{\varphi}}+\frac{\partial f}{\partial z}\hat{\boldsymbol{z}}$

(b) $\nabla\cdot\boldsymbol{A}=\frac{1}{r}\frac{\partial}{\partial r}(rA_r)+\frac{1}{r}\frac{\partial A_\varphi}{\partial\varphi}+\frac{\partial A_z}{\partial z}$

(c) $\nabla\times\boldsymbol{A}=\left(\frac{1}{r}\frac{\partial A_z}{\partial\varphi}-\frac{\partial A_\varphi}{\partial z}\right)\hat{\boldsymbol{r}}+\left(\frac{\partial A_r}{\partial z}-\frac{\partial A_z}{\partial r}\right)\hat{\boldsymbol{\varphi}}+\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\varphi)-\frac{\partial A_r}{\partial\varphi}\right)\hat{\boldsymbol{z}}$

(d) $\nabla^2 f=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 f}{\partial\varphi^2}+\frac{\partial^2 f}{\partial z^2}$

Spherical Coordinates (r,θ,φ)

(e) $\nabla f=\frac{\partial f}{\partial r}\hat{\boldsymbol{r}}+\frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\boldsymbol{\theta}}+\frac{1}{r\sin\theta}\frac{\partial f}{\partial\varphi}\hat{\boldsymbol{\varphi}}$

(f) $\nabla\cdot\boldsymbol{A}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta)+\frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$

(g) $\nabla\times\boldsymbol{A}=\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta A_\varphi)-\frac{\partial A_\theta}{\partial\varphi}\right)\hat{\boldsymbol{r}}+\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi}-\frac{\partial}{\partial r}(rA_\varphi)\right)\hat{\boldsymbol{\theta}}+\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta)-\frac{\partial A_r}{\partial\theta}\right)\hat{\boldsymbol{\varphi}}$

(h) $\nabla^2 f=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}$

5 Gaussian Integrals

Definite integral relations of Gaussian integrals

(a) $\int_0^\infty e^{-ax^2}\mathrm{d}x = \frac{1}{2}\left(\frac{\pi}{a}\right)^{1/2}$

(b) $\int_{-\infty}^\infty e^{-ax^2}\mathrm{d}x = \left(\frac{\pi}{a}\right)^{1/2}$

(c) $\int_{-\infty}^\infty e^{-ax^2}e^{-2bx}\mathrm{d}x = \left(\frac{\pi}{a}\right)^{1/2} e^{\frac{b^2}{a}}$ for $a > 0$

(d) $\int_{-\infty}^\infty xe^{-a(x-b)x^2}\mathrm{d}x = b\left(\frac{\pi}{a}\right)^{1/2}$

(e) $\int_{-\infty}^\infty x^2e^{-ax^2}\mathrm{d}x = \frac{1}{2}\left(\frac{\pi}{a^3}\right)^{1/2}$

(f) $$\int_0^\infty x^n e^{-ax^2}\mathrm{d}x = \begin{cases} \dfrac{1}{2}\Gamma\left(\dfrac{n+1}{2}\right)/a^{(n+1)/2} & a > 0 \\ \dfrac{(2k-1)!!}{2^{k+1}a^k}\sqrt{\dfrac{\pi}{a}} & n = 2k, a > 0 \\ \dfrac{k!}{2a^{k+1}} & n = 2k+1, a > 0 \end{cases}$$

6 Useful Fourier transforms

(a) $\mathcal{F}\left[\frac{\partial f}{\partial t}\right] = -i\omega\tilde{f}$

(b) $\mathcal{F}\left[\nabla f\right] = i\boldsymbol{k}\tilde{f}$