PHYS-E0414 Advanced Quantum Mechanics

Final exam, December 08, 2022, 13.00-16.00

You should answer in English unless you have special permission to use another language. You are free to use the lecture notes, books, the exercises, electronic devices, etc. (No communication allowed.) Please write your name, student number, study program, course code, and the date in all of your papers. There are 4 problems in this exam set which consists of 2 pages.

Exercise 1

Answer the following questions in your own words. No calculations are needed. Less than one page should suffice to answer all three questions:

a) This year's Nobel Prize in physics was partly given for experimental violations of Bell's inequality. What does it mean to violate Bell's inequality?

b) This year's Nobel Prize in physics also mentions quantum teleportation. Explain the concept of quantum teleportation.

c) For spin-1/2 particles, what do we call the operators that generate rotations of the spin?

Exercise 2

Consider three electrons, whose spins are in the state $|\Psi\rangle = |\uparrow_z\rangle_1 \left(|\uparrow_z\rangle_2|\downarrow_z\rangle_3 - |\downarrow_z\rangle_2|\uparrow_z\rangle_3\right)/\sqrt{2}$.

a) If the first two spins (1 and 2) are measured, what is the probability of finding them in the singlet state, $|\Psi_S\rangle = (|\uparrow_z\rangle_1|\downarrow_z\rangle_2 - |\downarrow_z\rangle_1|\uparrow_z\rangle_2)/\sqrt{2}$?

b) If the singlet state is found, what is the state of all three spins? For that state, what is the reduced density matrix of the third spin $\hat{\rho}_3$?

Exercise 3

In some materials, the spin and the momentum of electrons are coupled by the *spin-orbit* coupling. For example, an electron in a nanowire may be described by the Hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{H}_1 \text{ with } \hat{H}_0 = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_0^2\hat{x}^2 + B_z\hat{\sigma}_z \text{ and } \hat{H}_1 = \alpha\hat{p}\hat{\sigma}_z.$$

Here, the electron is trapped in a harmonic potential with frequency ω_0, a magnetic field is applied in the z-direction, and α denotes the strength of the spin-orbit coupling.

a) Argue that the eigenstates and eigenenergies of the Hamiltonian $\hat{H}_0$ are

$$|n, \pm\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0\rangle|\pm\rangle \text{ and } E^{(0)}_{n,\pm} = \hbar\omega_0(n + 1/2) \pm B_z \text{ with } n = 0, 1, 2, \ldots,$$

where $\hat{a}^\dagger, \hat{a}$ are the usual ladder operators, and $\hat{\sigma}_z|\pm\rangle = \pm 1|\pm\rangle$.

b) Using perturbation theory, find the eigenenergies $E_{0,\pm}$ of $\hat{H}$ up to second order in α. It may be useful to recall that $\hat{p} = i\sqrt{m\hbar\omega_0/2}(\hat{a}^\dagger - \hat{a})$.

c) Consider the unitary transformation, $\tilde{H} = \hat{U}\hat{H}\hat{U}^\dagger$, where $\hat{U} = e^{\frac{i}{\hbar}\alpha m\hat{x}\hat{\sigma}_z}$, and show that

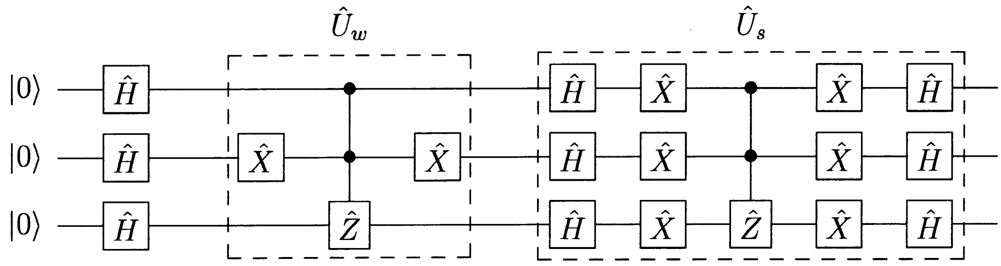
$$\tilde{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_0^2\hat{x}^2 + B_z\hat{\sigma}_z - \mathcal{E}_0,$$

where $\mathcal{E}_0$ is a constant that you should express in terms of m and α.

d) Use this expression to find the exact eigenvalues of $\hat{H}$ and compare with the result in b).

Exercise 4

The Grover search algorithm consists of repeated applications of the operator $\hat{U}_G = \hat{U}_s\hat{U}_w$ to the state $|s\rangle$, which we define below. An actual circuit implementation of the first step of the Grover algorithm on a quantum computer with 3 qubits may look as follows:



The $\hat{X}$-gate flips qubits, $\hat{X}|0\rangle = |1\rangle$ and $\hat{X}|1\rangle = |0\rangle$. The controlled $\hat{Z}$-gate adds a sign on the state $|1\rangle \to -|1\rangle$, if the two control-qubits (marked with circles) are both in the state $|1\rangle$.

a) Show that the first three Hadamard gates transform the input state to $|s\rangle = \hat{H}\otimes\hat{H}\otimes\hat{H}|000\rangle$ with $|s\rangle = (|000\rangle + \ldots + |101\rangle + |110\rangle + |111\rangle)/\sqrt{2}^3$. Show also that $\hat{H}\otimes\hat{H}\otimes\hat{H}|s\rangle = |000\rangle$.

b) Show that $\hat{U}_w|s\rangle = (|000\rangle + \ldots - |101\rangle + |110\rangle + |111\rangle)/\sqrt{2}^3$, where $\hat{U}_w$ is the first operator in the circuit indicated with a dashed box. Argue also that $\hat{U}_w|101\rangle = -|101\rangle$ and $\hat{U}_w|\overline{101}\rangle = |\overline{101}\rangle$, where $\overline{101}$ is any other bit string than 101.

c) The operator $\hat{U}_w$ *marks* the state $|101\rangle$ by adding a minus sign. Draw a circuit that would mark the state $|010\rangle$ and explain how it works. Draw a circuit that marks both states.

d) We now analyze the second operator $\hat{U}_s$ indicated with a dashed box in the circuit. Show that the operator acts as $\hat{U}_s|s\rangle = -|s\rangle$ and $\hat{U}_s|\overline{s}\rangle = |\overline{s}\rangle$ for any state $|\overline{s}\rangle$ that is orthogonal to $|s\rangle$, meaning that $\langle\overline{s}|s\rangle = 0$. Use these properties to show that $\hat{U}_s = -(2|s\rangle\langle s| - 1)$.

End of exam set