PHYS-E0415 Statistical Mechanics D – Final exam

The exam consists of three exercises each worth 6 course points. Please be sure to write your name, student number, course code and date to all of your solution papers. No calculators allowed.

1. Answer with a couple of sentences (6 p.)

(a) One of the important properties of random walks (RW) (in many applications) are the properties related to reaching site X from some starting point (say Y). A classical question related to this is: starting from origo (assume we have a RW in discrete time/lattice) what is the probability for an unbiased random walk to eventually return to the origo (same point)? (1 point for any reasonable argument). Let us now take the 2D case, where the RW moves in the discrete set of integers (x, y) one step in x or y (negative or positive). What is the probability in this case? (1 point). You may substitute with an argument for the distribution of times in the 1D case if you want.

(b) What happens if you approach the Ising critical point (say by tuning the temperature towards T_c) from above/below? (2 points).

(c) In the last lecture, we discussed how to create a (self-organized) critical system from a model of activated random walkers. How did we do this (qualitatively) and what happens, if we add bulk dissipation to the system? (2 points).

2. Life and the heat death of the Universe (6 p.)

Freeman Dyson discusses how living things might evolve to cope with the cooling and dimming we expect during the heat death of the Universe. Dyson ignores the survival and proliferation issues; he's interested in getting a lot of thinking in before the Universe ends. He presumes that an intelligent being generates a fixed entropy ΔS per thought.

Energy needed per thought. Assume that the being draws heat Q from a hot reservoir at T_1 and radiates it away to a cold reservoir at T_2.
(a) What is the minimum energy Q needed per thought, in terms of ΔS and T_2? You may take T_1 very large Related formulae: $\Delta S = Q_2/T_2 - Q_1/T_1$; $Q_1 - Q_2 = W$ (energy is conserved). (1 point)

Time needed per thought to radiate energy. Dyson shows, using theory not important here, that the power radiated by our intelligent-being-as-entropy-producer is no larger than CT_2^3, a constant times the cube of the cold temperature.
(b) Write an expression for the maximum rate of thoughts per unit time $\mathrm{d}H/\mathrm{d}t$ (the inverse of the time Δt per thought), in terms of ΔS, C, and T_2. (1 point)

$-\frac{9}{64} + \frac{64}{64} = \frac{55}{64}$ $t^{-\frac{3}{8}+1}$ $-\frac{3}{8} + \frac{8}{8} = \frac{5}{8}$

Number of thoughts for an ecologically efficient being. Our Universe is expanding; the radius R grows roughly linearly in time t. The microwave background radiation has a characteristic temperature $\Theta(t) \sim R^{-1}$ which is getting lower as the Universe expands; this red-shift is due to the Doppler effect. An ecologically efficient being would naturally try to use as little heat as possible, and so wants to choose T_2 as small as possible. It cannot radiate heat at a temperature below $T_2 = \Theta(t) = A/t$.
(c) How many thoughts H can an ecologically efficient being have between now and time infinity, in terms of ΔS, C, A, and the current time t_0? (2 points)

Time without end: greedy beings. Dyson would like his beings to be able to think an infinite number of thoughts before the Universe ends, but consume a finite amount of energy. He proposes that his beings need to be profligate in order to get their thoughts in before the world ends; he proposes that they radiate at a temperature $T_2(t) \sim t^{-3/8}$ which falls with time, but not as fast as $\Theta(t) \sim t^{-1}$.
(d) Show that with Dyson's cooling schedule, the total number of thoughts H is infinite, but the total energy consumed U is finite. (2 points)

3. Molecular zipper (6 p.)

Consider a system of N adjacent links forming a molecular zipper. The state of every link is either closed or open, and a link can be open only if its right-side neighbour is open (See Fig. 1). The energy cost of opening a link is ε and the degeneracy of an open link is $g > 1$. On the other hand, keeping a link closed has a cost of 0.

(a) Formulate an expression for the free energy $F = U - TS$ of the system with $n \leq N$ open links. Sketch a plot of $F(T)$ for n ranging from 0 to N. (2 points)

(b) With the results from (a), argue that there is a phase transition at some $T = T_c$. What is T_c? What is the average energy $\langle U \rangle$? Is the transition of first or second order? (4 points)

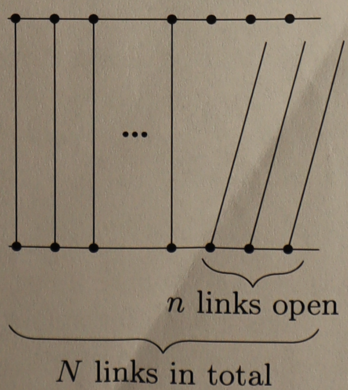

Figure 1: A schematic figure of the molecular zipper for exercise 3.