Department of Mathematics and Systems Analysis
Aalto University, School of Science
Metsalo

Mat-1.1620 Mathematics 2

3rd midterm exam, 13 May 2013

Please fill in clearly *on every sheet* the data on you and the examination. On *Examination code* mark course code, title and text mid-term or final examination. Degree Programmes are ARK, AUT, BIO, EST, ENE, GMA, INF, KEM, KTA, KON, MAR, MTE, PUU, RRT, TFM, TIK, TLT, TUO, YYT.

Calculators are not allowed. The examination time is 3 hours.
Ask, if You suspect typos in the text!

1. a) When changing variables in a triple integral by

$$\vec{r}(u,v,w) = x(u,v,w)\vec{\imath} + y(u,v,w)\vec{\jmath} + z(u,v,w)\vec{k},$$

the volume element $dV = dxdydz$ is replaced by $|\frac{\partial(x,y,z)}{\partial(u,v,w)}|dudvdw$, where $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ is the Jacobian of the transformation and can be interpreted as a scale factor.
Show that when changing from rectangular to spherical coordinates by $x(\rho,\phi,\theta) = \rho\sin\phi\cos\theta$, $y(\rho,\phi,\theta) = \rho\sin\phi\sin\theta$, $z(\rho,\phi,\theta) = \rho\cos\phi$, the Jacobian becomes $\frac{\partial(x,y,z)}{\partial(u,v,w)} = \rho^2\sin\phi$.
(This is the reason why $dV = dxdydz$ is not replaced by $d\rho d\phi d\theta$ but by $\rho^2\sin\phi d\rho d\phi d\theta$ when using spherical coordinates.)
b) Use spherical coordinates to calculate $\iiint_W xyz^2dxdydz$, where $W = \{(x,y,z) \in \mathbf{R}^3 | x, y \geq 0, x^2+y^2+z^2 \leq 1\}$ is a quarter of the unit ball.

2. $\vec{F}(x,y) = \frac{-y}{x^2+y^2}\vec{\imath} + \frac{x}{x^2+y^2}\vec{\jmath}$, where $(x,y) \in \mathbf{R}^2 \setminus \{(0,0)\}$.
a) Show that $\nabla \bullet \vec{F} \equiv 0$ in $\mathbf{R}^2\setminus\{(0,0)\}$.
b) Show that $\nabla \times \vec{F} \equiv \vec{0}$ in $\mathbf{R}^2\setminus\{(0,0)\}$.
c) Calculate $\oint_C \vec{F} \bullet d\vec{r}$, where C is the unit circle in the xy-plane, taken counterclockwise.

3. Calculate the mass of the triangle $S = \{(x,y,z) \in \mathbf{R}^3 | x + 2y + 3z = 6, x, y, z \geq 0\}$ in the figure below, if the area-density at the point $(x,y,z) \in S$ is given by $g(x,y,z) = x+y+z$.

4. a) Show that the differential equation $(2y + xy)dx + 2xdy = 0$ (which also can be written as $2y(x) + x \cdot y(x) + 2x \cdot y'(x) = 0$) is not exact, but can be made exact by multiplying by the integrating factor $\mu(x,y) = \frac{1}{xy}$.
b) Solve this differential equation with the initial condition $y(1) = 2$.