Department of Mathematics and Systems Analysis
Aalto University, School of Science

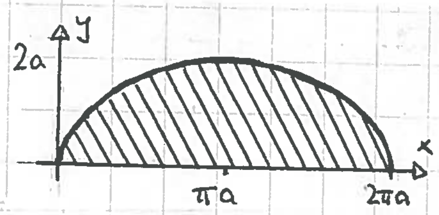
Metsalo

Mat-1.1620 Mathematics 2

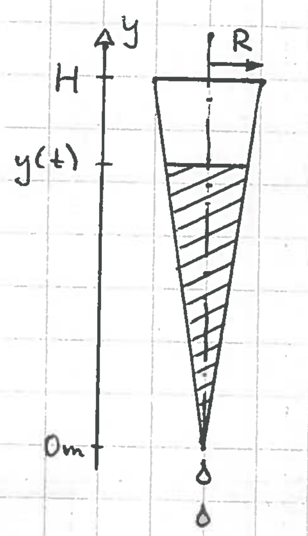
Final Exam, 26 February 2014

Please fill in clearly *on every sheet* the data on you and the examination. On *Examination code* mark course code, title and text mid-term or final examination. Degree Programmes are ARK, AUT, BIO, EST, ENE, GMA, INF, KEM, KTA, KON, MAR, MTE, PUU, RRT, TFM, TIK, TLT, TUO, YYT.

Calculators are not allowed. The examination time is 4 hours.
Ask, if You suspect typos in the text!



1. One arch of the cycloid can be given as $x(t) = a(t - \sin t)$, $y(t) = a(1 - \cos t), 0 \le t \le 2\pi, a > 0$. Calculate the area of the region above the x-axis but under one arch of the cycloid.



2. When a tank is emptied by gravity through a hole in the bottom, the height $y(t)$ of the liquid satisfies the separable ordinary differential equation (ODE) $A(y) \cdot dy/dt = -k \cdot \sqrt{y(t)}$, where $A(y)$ is the cross-sectional area of the tank at height y and k is a constant.
   A tank in the shape of a right circular cone with height H and radius R is standing on its tip. When it is filled with water, it gets emptied in time T through a hole at the tip. Set up the differential equation for the height and determine $y(t)$ for $t \in [0s, T]$. Determine also the time it takes for the height of the water to sink from H to $H/2$ (when only 1/8 of the water remains).

3. For rectangular coordinates x, y and polar coordinates r, θ in the plane we have $x = r\cos\theta, y = r\sin\theta \Rightarrow \tan\theta = y/x$ (when $x \neq 0$) and $r^2 = x^2 + y^2$.
   Show that $\frac{\partial x}{\partial \theta} \cdot \frac{\partial \theta}{\partial x} = \frac{\partial y}{\partial r} \cdot \frac{\partial r}{\partial y}$.

4. Let $\vec{F}(x, y, z) = \frac{-y}{x^2+y^2}\vec{\imath} + \frac{x}{x^2+y^2}\vec{\jmath} + z^2\vec{k}$. Calculate $\oint_c \vec{F} \bullet d\vec{r}$, where C is the closed curve starting at $(x, y, z) = (1, 0, 0)$, going along the helix $\vec{r}(t) = \cos t\vec{\imath} + \sin t\vec{\jmath} + t\vec{k}$ to $(1, 0, 2\pi)$ and then back to $(1, 0, 0)$ along a straight line.

5. $\nabla = \vec{\imath}\frac{\partial}{\partial x} + \vec{\jmath}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$. ∇ is not an ordinary vector, so rules that hold for ordinary vectors need not hold for ∇.
   a) For vectors $\vec{a}, \vec{b} \in \mathbf{R}^3$ we have that $\vec{b} \bullet (\vec{a} \times \vec{b}) = 0$. Give a vector field $\vec{G}(x, y, z)$ of class $C^1(\mathbf{R}^3)$ such that $\vec{G} \bullet (\nabla \times \vec{G}) = \vec{G} \bullet (curl(\vec{G})) \not\equiv 0$ (not identically equal to 0; it is OK if it is $= 0$ at some points). Calculate also $\vec{G} \bullet (\nabla \times \vec{G})$ at some point where $\vec{G} \bullet (\nabla \times \vec{G}) \neq 0$.
   b) For vectors $\vec{a}, \vec{b} \in \mathbf{R}^3$ we have that $\vec{a} \bullet (\vec{a} \times \vec{b}) = 0$. Show that for vector fields $\vec{H}(x, y, z)$ of class $C^2(\mathbf{R}^3)$ we have $\nabla \bullet (\nabla \times \vec{H}) = div(curl(\vec{H})) \equiv 0$. (Sometimes a rule for ordinary vectors *appears* to have a corresponding rule involving ∇.)

Useful (?) formulas:
$\cos^2 t + \sin^2 t = 1$, $\cos^2 t = (1 + \cos(2t))/2$, $\sin^2 t = (1 - \cos(2t))/2$,
$\sin(2t) = 2\sin t\cos t$, $\cos(2t) = \cos^2 t - \sin^2 t = 2\cos^2 t - 1 = 1 - 2\sin^2 t$.