Matrix Algebra
MS-A0001
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Exam, 24.2.2023

Every question carries an equal weight, similarly every part of a question carries an equal weight, unless otherwise specified.

PROBLEM 1 A *commutator* of matrices A and B is defined as

$$C = [A, B] = A\,B - B\,A.$$

a) Discuss, why the definition is meaningful if and only if both A and B are square matrices of equal size. b) Evaluate the commutator, when

$$A = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ -2 & 0 \end{pmatrix}.$$

PROBLEM 2 Using only the definition of the matrix product and the transpose, show that when the product is defined, then

$$(A\,B)^T = B^T\,A^T.$$

PROBLEM 3 Find the LU-decomposition of

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

PROBLEM 4 Find all solutions of $Ax = b$, where

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -1 & \alpha \\ -1 & 2 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 2 \\ \beta \end{pmatrix},$$

$\alpha, \beta \in \mathbb{R}$.

PROBLEM 5 Show that every orthogonal 2×2-matrix can be expressed in either one of the two forms:

$$\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix}.$$

PROBLEM 6 (a) Let A be an invertible square matrix such that (λ, x) is an eigenpair, $\lambda \neq 0$, $x \neq o$. Show that $(1/\lambda, x)$ is an eigenpair of A^{-1}.
(b) Let the matrix A have exactly two eigenvalues $\lambda_1 = 1$, $\lambda_2 = 1/2$, and the corresponding eigenvectors $v_1 = (1,1)^T$, $v_2 = (-1,1)^T$. Find the limit $\lim_{k\to\infty} A^k$.