MS-C2111 Stochastic processes
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Exam
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The exam consists of 4 problems, each worth 6 points. You are **not** allowed to use a calculator in the exam. However, you are allowed to use a **handwritten memory aid sheet** of size A4, with text only on one side and with your name and student number in the upper right corner.

1 Jack is lost in a snowstorm. He moves on the state space $\{1, 2, 3, 4, 5, 6\}$ according to a discrete-time Markov chain with transition matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0.2 & 0.2 & 0 & 0.2 & 0.2 & 0.2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0.3 & 0.2 & 0 & 0.5 \\ 0 & 0 & 0 & 0.2 & 0.8 & 0 \end{bmatrix}$$

Initially Jack is on the top of a mountain (state 3) when the snowstorm hits. He hopes to eventually reach home (state 1). He also hopes to visit his sauna (state 5) while avoiding a bear's den (state 4).

Justify your answers to the following questions.

(a) **(2p.)** What is the probability that Jack eventually reaches home?

(b) **(1p.)** What is the probability that Jack never visits his sauna?

(c) **(2p.)** What is the expected duration until Jack ends up in $\{1, 4\}$?

(d) **(1p.)** Is it possible that Jack remains wandering in the snowstorm forever, without ever reaching home or the bear's den?

2 Developers of two computer algorithms "muGo" and "nuGo" want to investigate which one is stronger in the game of go. They do so by letting muGo and nuGo play against each other repeatedly until they are happy with their conclusion. At each round, muGo wins with probability 0.8 independently from other games. Denote the number of games after $t = 0, 1, 2, \ldots$ rounds won by muGo and nuGo by M_t and N_t respectively, and let $X_t = M_t - N_t$.

Justify your answers to the following questions.

(a) **(2p.)** Model the process $X = (X_t)_{t \in \mathbb{N}}$ as a Markov chain. What is its state space? Write down its transition matrix and draw its transition diagram.

(b) **(2p.)** Does X have an invariant distribution? If yes, how many?

(c) **(2p.)** The developers decide to let muGo and nuGo to compete in a tournament. The first algorithm to win two consecutive games is declared as winner. What is the probability that nuGo wins the tournament?

3 The Otaniemi library has two copies of Kallenberg's classic textbook. Students wishing to borrow the book arrive in the library at random exponentially distributed times with an average rate of 0.4 students per day. If upon such a student's arrival there is a copy of the book available, the student will borrow the book for a random time of exponentially distributed duration with an average of 10 days. If there is no copy available, the arriving student will borrow some other (less interesting) book. The library is open all the time, and the arrival times and borrow durations are mutually independent.

Justify your answers to the following questions.

(a) **(1p.)** Let X_t be the number of copies of Kallenberg's books borrowed from the library at time $t \geq 0$, where $X_0 = 0$. Justify why $(X_t)_{t \in \mathbb{R}_+}$ is a Markov process. Determine its state space and generator matrix, and draw its transition diagram.

(b) **(2p.)** What is the probability at the statistical equilibrium that when a student arrives to borrow Kallenberg's book, there is a copy of it available in the library?

(c) **(2p.)** If both copies of Kallenberg's book are borrowed from the library at a given time t, what is the probability that at least one student will arrive in the library to try to borrow the book before either of the copies has been returned to the library?

(d) **(1p.)** Explain how to compute the probability that, given that on Monday morning at 9 a.m. both copies of Kallenberg's book are available in the library, on the following Tuesday morning at 9 a.m., both copies are unavailable.

4 A new disease spreads in the city of Espoo. Epidemiologists have decided to model the epidemic using a branching process $X = (X_t)_{t \in \mathbb{N}}$ on an infinite population size.

In the morning of day zero, one person is infectious and the rest are susceptible for the disease. During each day, every infectious person contacts two susceptible persons, and in the evening the infectious person recovers and becomes immune. Each contact is infectious with probability 2/3, independently of others. If a contact is infectious, then the targeted susceptible person gets an infection and becomes infectious in the morning of the next day. Justify your answers to the following questions.

(a) **(1p.)** Write down the probability generating function for the offspring distribution of the initially infected person.

(b) **(1p.)** Compute the expected number of people who directly receive the infection from the initially infected person.

(c) **(1p.)** What is the probability that the epidemic eventually dies out, so that on some day there will be no infectious people in the population?

(d) **(1p.)** Find $\lambda \in \mathbb{R}$ for which the process $(M_t)_{t \in \mathbb{N}} := (\lambda^{-t} X_t)_{t \in \mathbb{N}}$ is a martingale.

(e) **(2p.)** One evening 90 susceptible citizens are identified to have been in contact with an infectious person. They are tested using an 80% reliable test: for infected person the test is positive with probability 0.8, and for non-infected person it is positive with probability 0.2. The test results are independent from each other. Out of the 90 tests, 30 turned out positive. Given this information, what is the probability that the epidemic eventually dies out?