Exam

17.04.2023

- You need to justify all of your answers.
- You can use a non-graphing calculator.
- You can use one two-sided cheat sheet (maximal DIN A4).

Problem 1 (20 points)

Consider the following linear program.

$$\begin{aligned} \min\ & 2x_1 + x_2 \\ \text{s.t. } & x_1 + x_2 \geq 6 \\ & 2x_1 - x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

a) (10 points) Bring the program into standard form and find a feasible basic solution.

b) (10 points) Compute an optimal solution using the simplex algorithm.

Problem 2 (20 points)

A factory produces three products, A, B, and C, from two ingredients I_1 and I_2. To produce one ton (t) of product A, 0.5t of I_1 and 0.6t of I_2 are needed. For one ton of product B, 0.7t of I_1 and 0.5t of I_2 are needed. For one ton of product C, 0.6t of I_1 and 0.6t of I_2 are needed. Of ingredient I_1, 200t are available and of ingredient I_2, 300t are available. Additionally, more of ingredient I_1 can be purchased for 100€ per ton. Products A, B, and C can be sold for 200€, 250€ and 220€ per ton, respectively. We assume that all amount of the products A, B, C that is produced will be sold. The goal is to maximise the revenue, i.e., the income minus the expenses for purchasing ingredient I_1.
Hint: *You do not need to solve this problem.*

a) (10 points) Model the problem as a linear program.

b) (5 points) Suppose that at most two products can be produced. Derive a corresponding model based on the model from a).
Hint: *The new model does not have to be a linear program.*

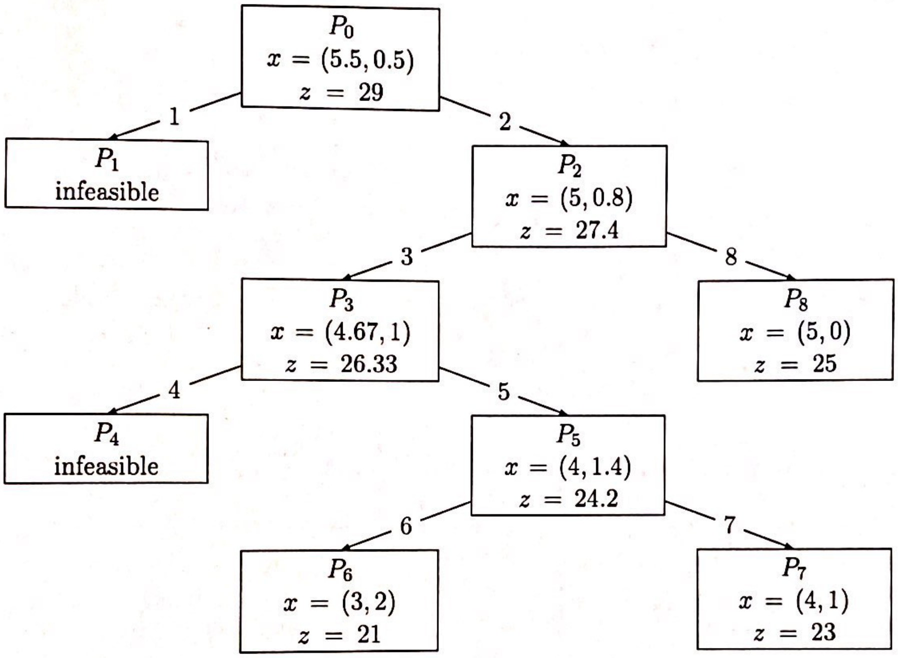
c) (5 points) Suppose that the price of product A depends on the amount produced and sold. For product A, the price per ton reduces by 0.5€ for each ton sold. Thus, 5t would be sold for 197.5€ per ton and 50t would be sold for 175€ per ton. How do you have to adapt the original objective function to accommodate for this change?
Hint: *The new objective function does not have to be linear.*

Problem 3 (15 points)

The problem

$$\begin{aligned} \max\ & 5x_1 + 3x_2 \\ \text{s.t. } & 4x_1 + 2x_2 \le 23 \\ & 3x_1 + 5x_2 \le 19 \\ & x_1, x_2 \ge 0 \\ & x_1, x_2 \in \mathbb{Z} \end{aligned}$$

is solved by the branch-and-bound method resulting in the following branch-and-bound tree. The subproblems are solved in the order P_0, P_1, …, P_8.



a) (10 points) Give the additional constraints which correspond to branching decisions on the edges 1 to 8. What is the optimal solution of the problem?

b) (5 points) Identify a different order to solve the subproblems such that not all subproblems need to be solved, i.e., such that at least one branch can be pruned.

Problem 4 (20 points)

Consider the following problem

$$\begin{aligned} \min\ & (1 - x_1)^2 + 3x_2^2 \\ \text{s.t. } & x_1^2 - 4 \le 0 \\ & x_1 + 3x_2 = 4. \end{aligned}$$

Formulate the KKT conditions and find a point that satisfies them. Are these conditions sufficient for global optimality?

Problem 5 (15 points)

Consider the following barrier problem

$$B_\rho\colon \min\ f(x_1, x_2) = x_1 x_2^2 - (x_2 - 3)^2 - \rho \ln(x_1) - \rho \ln(x_2)$$

with $\rho > 0$ describing the accuracy of the barrier term.
Hint: *ρ can be handled as a fixed parameter in this exercise.*

a) (5 points) Formulate an optimisation problem (P) such that B_ρ is the corresponding barrier problem.

b) (10 points) Suppose for $x^0 \in \mathbb{R}^2$ that Δx solves

$$\nabla f(x^0) + H(x^0)\Delta x = 0$$

where $\nabla f(x^0)$ is the gradient of f at x^0 and $H(x^0)$ the Hessian of f at x^0. Is $x^1 = x^0 + \Delta x$ guaranteed to satisfy the necessary optimality conditions of B_ρ? If not, how can you use x^1 to find a better approximation of an optimal solution of B_ρ?

Problem 6 (10 points)

Consider two integer programs (P) and (D).

$$(P): \max\ c^\top x \quad \text{s.t. } Ax \le b,\ x \ge 0,\ x \in \mathbb{Z}^n$$

$$(D): \min\ b^\top y \quad \text{s.t. } A^\top y \ge c,\ y \ge 0,\ y \in \mathbb{Z}^m$$

Show that weak duality holds, i.e., any feasible solution x of (P) and any feasible solution y of (D) satisfy

$$c^\top x \le b^\top y.$$