Aalto University
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Course Exam, Wednesday 19.04.2023, 09:00 - 12:00
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Differential and Integral Calculus 3, MS-A0311

No calculators or tables of formulas allowed. **See backside for the allowed collection of formulas!**

Motivate your answers. Only giving answers gives no points. **The course exam** consists of the four exercises with best result out of **exercise 1,2,3,4, and 5**. **The exam** consists of **exercise 1,2,3,4, and 5**. If you prefer you can do all five exercises and I will evaluate using *"the result on the course exam + points given during the course"* or *"the result on the exam"*. The alternative giving the best grade will be used.

(1) Let $\mathbf{F}(x,y) = (1, -xy)$. Determine all integral curves. Sketch the integral curve through the point $(x,y) = (0,1)$. (6p)

(2) Let $\mathcal{S}$ be the surface given by

$$\mathcal{S} = \left\{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \leq 4 \text{ and } z = (x^2+y^2)/4\right\}.$$

(a) Calculate

$$\iint_{\mathcal{S}} 1 \, dS.$$

(3p)

(b) Let $\mathbf{F}(x,y,z) = (x, y, -3z^2)$ and $\mathbf{n}$ be the unit normal field to $\mathcal{S}$ pointing downwards. Calculate

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n} \, dS.$$

(3p)

(*Hint:* Polar coordinates help here.)

(3) Let $\mathbf{F}(x,y,z) = (x+y, x+y, x+y)$, and

$$\mathcal{S}_R = \left\{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = R^2\right\}$$

for $R > 0$. Let $\mathbf{n}$ be the unit normal field to $\mathcal{S}_R$ pointing away from the origin. Calculate

$$\oiint_{\mathcal{S}_R} \mathbf{F} \cdot \mathbf{n} \, dS.$$

(6p)

(4) Let $a, b,$ and c be constants and $\mathbf{F}(x,y,z) = (x + axy + z, x^2 - y^2, bx - cz)$.
(a) Determine $a, b,$ and c so that $\operatorname{div} \mathbf{F} = 0$ and $\operatorname{Curl} \mathbf{F} = (0,0,0)$. (2$p$)
(b) With the values of $a, b,$ and c from (a), construct a potential function $\phi(x,y,z)$ for $\mathbf{F}$ such that $\phi(1,2,3) = 0$. (2p)
(c) With the values of $a, b,$ and c from (a), show that $\int_\gamma \mathbf{F} \cdot d\mathbf{r} = 0$ for any piecewise smooth simple closed curve γ. (2p)

(5) Let γ be the half-circle given by $\gamma(t) = (\cos t, \sin t)$, $0 \leq t \leq \pi$. Let

$$\mathbf{F}(x,y) = (1 - 2y^3, 2x^3 + e^{y^2}).$$

Calculate $\int_\gamma \mathbf{F} \cdot d\mathbf{r}$. (*Hint:* Notice that γ is not closed. Try to find a suitable curve σ so that $\gamma + \sigma$ is closed and then use Green's Theorem.) (6p)

Good luck!

Useful theorems and formulas:

- Green's Theorem:

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \oint_\gamma \mathbf{F} \cdot d\mathbf{r}$$

- Stokes's Theorem:

$$\iint_S (\operatorname{Curl} \mathbf{F}) \cdot \mathbf{n}\, dS = \oint_\gamma \mathbf{F} \cdot d\mathbf{r}$$

- Gauss's Theorem:

$$\iiint_D (\operatorname{div} \mathbf{F})\, dV = \oiint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

- Gradient in a orthogonal curvilinear coordinate system $[\hat{u}, \hat{v}, \hat{w}]$,

$$\nabla f = \frac{1}{h_u}\frac{\partial f}{\partial u}\hat{u} + \frac{1}{h_v}\frac{\partial f}{\partial v}\hat{v} + \frac{1}{h_w}\frac{\partial f}{\partial w}\hat{w}$$

- Divergence in a orthogonal curvilinear coordinate system $[\hat{u}, \hat{v}, \hat{w}]$,

$$\operatorname{div} \mathbf{F} = \frac{1}{h_u h_v h_w}\left(\frac{\partial}{\partial u}(F_u h_v h_w) + \frac{\partial}{\partial v}(F_v h_u h_w) + \frac{\partial}{\partial w}(F_w h_u h_v)\right)$$

- Curl in a positively oriented orthogonal curvilinear coordinate system $[\hat{u}, \hat{v}, \hat{w}]$,

$$\operatorname{Curl} \mathbf{F} = \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u\hat{u} & h_v\hat{v} & h_w\hat{w} \\ \partial/\partial u & \partial/\partial v & \partial/\partial w \\ h_u F_u & h_v F_v & h_w F_w \end{vmatrix}$$