Question 1 *(Concepts)*

Describe briefly:

a) What is a RLC resonator and how is the quality factor defined and calculated?

b) What are the Josephson relations? What is the Josephson inductance?

c) What is a SQUID and how does it work?

d) What is a transmon qubit and how is it controlled and measured?

Question 2 *(LC circuits)*

Consider the circuit shown in Fig. 1, where we are interested in what happens with the signals of different frequencies present at the input.

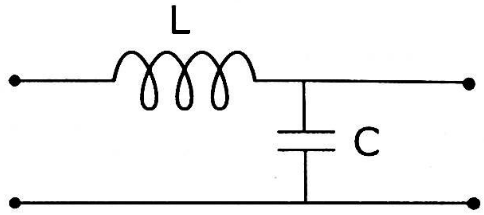

Figure 1 – A circuit consisting of an inductor and a capacitor.

(a) Suppose that you have a dc input. Is this transmitted and why? What if you add an ac-component, what do you expect to get at the output, and how would it depend on frequency? What is the capacitor doing for the ac-components?

(b) Analyze the output vs. input in terms of the ac impedances of the inductor and capacitor. Sketch a plot of the results.

Question 3 *(Superconductivity)*

In the lecture we derived the first London equation as $\mathbf{J}_s = -\frac{n_s q^{*2}}{m^*}\mathbf{A}$, and we also used the notation $\lambda_L = \sqrt{m^*/(\mu_0 n_s q^{*2})}$.

(a) Briefly explain the meaning of the quantities in the formula above. Was there any quantum-mechanical concept used in deriving it, or it is a purely classical result?

(b) By using Maxwell's equation $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}_s$, derive a second-order differential equation for the magnetic field. Here you may find useful the identity $\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$ and you may work in the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$. Remember that $\mathbf{B} = \nabla \times \mathbf{A}$.

(c) Consider the configuration in Fig. 2 where a magnetic field $\mathbf{B}(z) = (B(z), 0, 0)$ along the x-direction is present outside the superconductor. Solve the equation at (b) for this field inside the superconductor. We also have the other Maxwell equation for magnetic fields, $\nabla \cdot \mathbf{B} = 0$. Does your solution satisfy this equation?

(d) A MgB_2 superconducting sample was measure to have $\lambda_L = 150$ nm. By how much is the field reduced at a depth of 300 nm inside the superconductor?

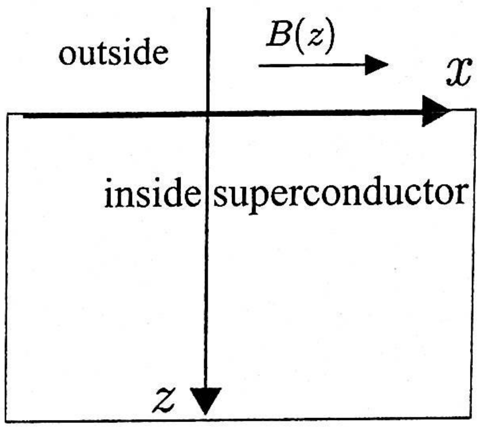

Figure 2 – Schematic of a magnetic field in the vicinity of a superconductor.

Question 4 *(Lagrangian and Hamiltonian of a superconducting circuit)*

Consider the following Lagrangian:

$$\mathscr{L}(\phi, \dot{\phi}) = \frac{1}{2}C_g(\dot{\phi} - V_g)^2 + \frac{1}{2}C_J\dot{\phi}^2 + E_J\cos\left(\frac{2\pi}{\Phi_0}\phi\right). \tag{1}$$

(a) What kind of quantum circuit does this Lagrangian represent? What are the generalized coordinates?

(b) Sketch a circuit diagram, labelling and defining all relevant parameters from the Lagrangian in the context of the physical quantum circuit.

(c) Set $V_g = 0$. Derive the Hamiltonian of the circuit using Legendre transformation (see the hint).

Hint: $\mathscr{H} = \sum_i \dot{x}_i p_i - \mathscr{L}$, where x_i and $\dot{x}_i$ are the generalized coordinates. $p_i = \frac{\partial \mathscr{L}}{\partial \dot{x}_i}$ is the conjugate momentum.

Question 5 *(Qubit measurement)*

Consider a qubit (an ideal two-level system) strongly coupled to a resonator for the measurement of the qubit state and the following is the Hamiltonian for the system in the dispersive limit of the Jaynes-Cummings model:

$$\hat{H}_{\text{disp}} = (\omega_r + \chi\hat{\sigma}_z)(\hat{a}^\dagger\hat{a} + \frac{1}{2}) + \frac{\omega_q}{2}\hat{\sigma}_z, \tag{2}$$

(a) What would be the frequencies of the resonator for qubit state $|0\rangle$ and $|1\rangle$?

(b) If there are no photons in the resonator, what is the expected qubit frequency? If there are n photons, what will be the frequency of the qubit?

(c) Describe briefly (best would be by using a schematic or a drawing for the measured data you expect) how the dispersive Jaynes-Cummings model can be used in practice to measure the state of a superconducting qubit.

Hint: For part (a), apply the Hamiltonian to the qubit basis states $|0\rangle$ and $|1\rangle$ and find what is happening to the coefficient of $(\hat{a}^\dagger\hat{a} + \frac{1}{2})$. Use $\hat{\sigma}_z|0\rangle = |0\rangle$ and $\hat{\sigma}_z|1\rangle = -|1\rangle$.

Question 6 *(Quantum gates)*

Consider the unitary operator:

$$\hat{U}_{\rm qq}(t) = \exp\left[-i\frac{g}{2}(\hat{\sigma}_x\otimes\hat{\sigma}_x + \hat{\sigma}_y\otimes\hat{\sigma}_y)t\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(gt) & -i\sin(gt) & 0 \\ 0 & -i\sin(gt) & \cos(gt) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}$$

(a) What physical process does this unitary operator correspond to? What is the significance of this process in the context of quantum computing?

(b) Consider the following input-output relations:

$$\hat{U}_{\rm qq}(t_{\rm g})\,|00\rangle = |00\rangle$$
$$\hat{U}_{\rm qq}(t_{\rm g})\,|01\rangle = -i\,|10\rangle$$
$$\hat{U}_{\rm qq}(t_{\rm g})\,|10\rangle = -i\,|01\rangle$$
$$\hat{U}_{\rm qq}(t_{\rm g})\,|11\rangle = |11\rangle\,.$$

Based on these relations, explain what sort of operation is being performed to the input states? What value of $t_{\rm g}$ corresponds to this operation? Use this value to write down the matrix form of the performed operation.

Usual definitions: $|00\rangle = |0\rangle\otimes|0\rangle$, $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Pauli matrices: $\hat{\sigma}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\hat{\sigma}_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$. Also $\hat{\mathbb{1}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the 2×2 identity matrix.