CIV-E4010 Finite Element Methods in Civil Engineering Examination, April 18, 2023 / Niiranen

This examination consists of 3 problems rated by the standard scale 1...6. Neither course material nor calculators are allowed – pen(cil) and paper only.

Problem 1

Let us consider the finite element method in the context of *structural mechanics* and the *theory of elasticity*.

(i) Briefly sketch the *linear Lagrange*-type *Timoshenko beam* finite element: (1) identify the number of nodes in one element; (2) list the degrees of freedom present at each node; (3) determine the size of both the local stiffness matrix and the force vector of the element?

(ii) Briefly sketch the *quadratic Lagrange*-type *Timoshenko beam* finite element: (1) identify the number of nodes in one element; (2) list the degrees of freedom present at each node; (3) determine the size of both the local stiffness matrix and the force vector of the element?

(iii) Briefly explain, preferably with a few formulae, what is meant by *numerical locking* in the context of *Timoshenko beam* elements.

Problem 2

Let us consider the finite element method in the context of *structural mechanics* and the *theory of elasticity*.

(i) The governing differential equation of the *linear buckling* problem of elastic beams following the *Euler–Bernoulli beam* theory can be written in the form

$$(EIw'')''(x) + Pw''(x) = 0,\ 0 < x < L.$$

Write down (without deriving) the corresponding weak form by assuming that both ends of the beam are simply supported: "*Find...such that...*".

(ii) Use one *Hermite*-type finite element for finding an approximate solution to the problem of item (i): (1) form the required finite element system equation; (2) form the characteristic equation of the problem.

For constructing the finite element equation system for the problem, there is no need to start from the basis functions: the *stiffness matrix* and *geometric stiffness matrix* of the element are given, respectively, as

$$
\boldsymbol{K} = \frac{EI}{L^3} \begin{pmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ & & & 4L^2 \end{pmatrix}
$$

$$
\boldsymbol{K}_g = \frac{1}{30L} \begin{pmatrix} 36 & 3L & -36 & 3L \\ & 4L^2 & -3L & -L^2 \\ & & 36 & -3L \\ & & & 4L^2 \end{pmatrix}.
$$

Problem 3

Let us consider the finite element method in the context of *structural mechanics* and the *theory of elasticity.*

(i) When solving the basic problems of *linear statics*, the resulting algebraic equation system can be written in the form $\boldsymbol{K}\boldsymbol{d} = \boldsymbol{f}$.

(1) Write down the equation system for the corresponding problem of *linear dynamics.* (2) Briefly explain the essential content of the system and its physical background. (3) Briefly explain the main differences in solving the equation systems of these two problem types (*linear statics* and *linear dynamics*).

(ii) For a conforming finite element method of the *Kirchhoff plate* problem, (with certain additional assumptions) the basic mathematical finite element error estimate is of the form

$$
\|w - w_h\|_2 \leq Ch^{k-1}|w|_{k+1}.
$$

Define and name the quantities, variables, indices and other notation appearing in the inequality, and shortly describe the information this estimate provides.

(iii) For isotropic materials, the moment vector of the *Kirchhoff plate* problem can be written as

$$
\boldsymbol{M}(\nabla w) = \boldsymbol{D}\boldsymbol{\kappa}(\nabla w)
$$

$$
\boldsymbol{D} = D \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{pmatrix}, \quad D = \frac{Et^3}{12(1-\nu^2)}
$$

$$
\boldsymbol{\kappa}(\nabla w) = \begin{pmatrix} -\partial^2 w/\partial x^2 \\ -\partial^2 w/\partial y^2 \\ -2\partial^2 w/\partial x \partial x) \end{pmatrix}.
$$

By utilizing the error estimate given in item (ii), write down an error estimate for the finite element approximation of the bending moment.