MS-A0001 Matrix Algebra Exam Solutions

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Problem 1 A commutator of matrices A and B is defined as

$$C = [A, B] = AB - BA$$

a) Discuss, why the definition is meaningful if and only if both A and B are square matrices of equal size. b) Evaluate the commutator, when

$$A = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ -2 & 0 \end{pmatrix}$$

Problem 1 solution

a) The definition of the commutator is meaningful if and only if both A and B are square matrices of equal size because the matrix multiplication is only defined for matrices where the number of columns in the first matrix is equal to the number of rows in the second matrix. In the commutator definition, we have two matrix multiplications: AB and BA.

For AB to be defined, the number of columns in A must be equal to the number of rows in B. Similarly, for BA to be defined, the number of columns in B must be equal to the number of rows in A. Since both A and B are square matrices, they have the same number of rows and columns. Therefore, the commutator definition is meaningful if and only if both A and B are square matrices of equal size.

b) Now, let's evaluate the commutator when

$$A = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ -2 & 0 \end{pmatrix}$$

First, we need to find the product of the matrices AB:

$$AB = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & 1 \end{pmatrix}$$

Next, we need to find the product of the matrices BA:

$$BA = \begin{pmatrix} 2 & 1 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -2 & 0 \end{pmatrix}$$

Now, we can find the commutator $C = AB - BA$:

$$C = \begin{pmatrix} 2 & 1 \\ 4 & 1 \end{pmatrix} - \begin{pmatrix} 3 & -1 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 6 & 1 \end{pmatrix}$$

So, the commutator of matrices A and B is:

$$C = \begin{pmatrix} -1 & 2 \\ 6 & 1 \end{pmatrix}$$

Problem 2 Using only the definition of the matrix product and the transpose, show that when the product is defined, then

$$(AB)^T = B^T A^T$$

Problem 2 solution

To show that $(AB)^T = B^T A^T$, we will use the definition of the matrix product and the transpose. Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Then, the product AB is an $m \times p$ matrix, and its (i, j)-th entry is given by:

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

Now, let's find the transpose of the product AB. The transpose of a matrix is obtained by interchanging its rows and columns. So, the (j, i)-th entry of $(AB)^T$ is given by:

$$(AB)^T_{ji} = (AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

Next, let's find the product of the transposes $B^T A^T$. The transpose of matrix A is an $n \times m$ matrix, and its (k, i)-th entry is given by:

$$A^T_{ki} = A_{ik}$$

Similarly, the transpose of matrix B is an $p \times n$ matrix, and its (j, k)-th entry is given by:

$$B^T_{jk} = B_{kj}$$

Now, let's find the (j, i)-th entry of the product $B^T A^T$:

$$(B^T A^T)_{ji} = \sum_{k=1}^{n} B^T_{jk} A^T_{ki} = \sum_{k=1}^{n} B_{kj} A_{ik}$$

Comparing the (j, i)-th entries of $(AB)^T$ and $B^T A^T$, we have:

$$(AB)^T_{ji} = \sum_{k=1}^{n} A_{ik} B_{kj} = \sum_{k=1}^{n} B_{kj} A_{ik} = (B^T A^T)_{ji}$$

Since the (j, i)-th entries of $(AB)^T$ and $B^T A^T$ are equal for all i and j, we can conclude that:

$$(AB)^T = B^T A^T$$

Problem 3 Find the LU-decomposition of

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

Problem 3 solution

To find the LU-decomposition of the given matrix A, we want to find two matrices L and U such that $A = LU$, where L is a lower triangular matrix and U is an upper triangular matrix.

Let's write the general form of the matrices L and U:

$$L = \begin{pmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{pmatrix}, \quad U = \begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{pmatrix}$$

Now, we want to find the entries of L and U such that $A = LU$. We can do this by equating the corresponding entries of the product LU and the matrix A.

We have:

$$LU = \begin{pmatrix} l_{11}u_{11} & l_{11}u_{12} & l_{11}u_{13} & l_{11}u_{14} \\ l_{21}u_{11} & l_{21}u_{12} + l_{22}u_{22} & l_{21}u_{13} + l_{22}u_{23} & l_{21}u_{14} + l_{22}u_{24} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + l_{33}u_{33} & l_{31}u_{14} + l_{32}u_{24} + l_{33}u_{34} \\ l_{41}u_{11} & l_{41}u_{12} + l_{42}u_{22} & l_{41}u_{13} + l_{42}u_{23} + l_{43}u_{33} & l_{41}u_{14} + l_{42}u_{24} + l_{43}u_{34} + l_{44}u_{44} \end{pmatrix}$$

Comparing the entries of LU and A, we get the following equations:

$$\begin{aligned}
l_{11}u_{11} &= 2 \\
l_{11}u_{12} &= -1 \\
l_{21}u_{11} &= -1 \\
l_{21}u_{12} + l_{22}u_{22} &= 2 \\
l_{11}u_{13} &= 0 \\
l_{21}u_{13} + l_{22}u_{23} &= -1 \\
l_{31}u_{11} &= 0 \\
l_{31}u_{12} + l_{32}u_{22} &= -1 \\
l_{31}u_{13} + l_{32}u_{23} + l_{33}u_{33} &= 2 \\
l_{11}u_{14} &= 0 \\
l_{21}u_{14} + l_{22}u_{24} &= 0 \\
l_{31}u_{14} + l_{32}u_{24} + l_{33}u_{34} &= -1 \\
l_{41}u_{11} &= 0 \\
l_{41}u_{12} + l_{42}u_{22} &= 0 \\
l_{41}u_{13} + l_{42}u_{23} + l_{43}u_{33} &= -1 \\
l_{41}u_{14} + l_{42}u_{24} + l_{43}u_{34} + l_{44}u_{44} &= 2
\end{aligned}$$

Solving these equations, we find the entries of L and U:

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{pmatrix}$$

So, the LU-decomposition of the matrix A is given by the matrices L and U above.

Problem 4 Find all solutions of $Ax = b$, where

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -1 & \alpha \\ -1 & 2 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 2 \\ \beta \end{pmatrix}$$

$$\alpha, \beta \in \mathbb{R}$$

Problem 4 solution

To find all solutions of the system $Ax = b$, we can use the Gaussian elimination method. First, let's write the augmented matrix for the system:

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & -1 & \alpha & 2 \\ -1 & 2 & 1 & \beta \end{array}\right)$$

Now, let's perform Gaussian elimination:
1. Subtract 2 times the first row from the second row:

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & \alpha - 4 & 0 \\ -1 & 2 & 1 & \beta \end{array}\right)$$

2. Add the first row to the third row:

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & \alpha - 4 & 0 \\ 0 & 1 & 3 & \beta + 1 \end{array}\right)$$

3. Subtract the second row from the third row:

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & \alpha - 4 & 0 \\ 0 & 0 & 7 - \alpha & \beta + 1 \end{array}\right)$$

Now, let's analyze the possible cases for the values of α and β:

Case 1: $\alpha \neq 7$

In this case, the last row of the matrix represents the equation $(7 - \alpha)z = \beta + 1$. Since $\alpha \neq 7$, we can divide both sides of the equation by $(7 - \alpha)$ to get:

$$z = \frac{\beta + 1}{7 - \alpha}$$

Now, we can use back-substitution to find the values of y and x:

$$y = \frac{(\alpha - 4)z}{1} = (\alpha - 4)z$$

$$x = 1 + y - 2z = 1 + (\alpha - 4)z - 2z$$

So, the solution for this case is:

$$x = 1 + (\alpha - 6)z, \quad y = (\alpha - 4)z, \quad z = \frac{\beta + 1}{7 - \alpha}$$

Case 2: $\alpha = 7$

In this case, the last row of the matrix represents the equation $0z = \beta + 1$. If $\beta = -1$, the system has infinitely many solutions, and we can express the solution in terms of a free variable z:

$$x = 1 + y - 2z, \quad y = 3z, \quad z \in \mathbb{R}$$

If $\beta \neq -1$, the system has no solution, as the last row represents an inconsistent equation.

In conclusion, the system $Ax = b$ has a unique solution for $\alpha \neq 7$, given by:

$$x = 1 + (\alpha - 6)z, \quad y = (\alpha - 4)z, \quad z = \frac{\beta + 1}{7 - \alpha}$$

For $\alpha = 7$, the system has infinitely many solutions if $\beta = -1$, given by:

$$x = 1 + y - 2z, \quad y = 3z, \quad z \in \mathbb{R}$$

Otherwise, the system has no solution.

Problem 5 Show that every orthogonal 2×2-matrix can be expressed in either one of the two forms:

$$\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \text{ or } \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix}$$

Problem 5 solution

Let A be an orthogonal 2×2 matrix. Then, by definition, we have $A^T A = I$, where I is the identity matrix. Let's write the general form of the matrix A:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Now, let's find the transpose of A:

$$A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

Now, let's compute the product $A^T A$:

$$A^T A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix}$$

Since A is orthogonal, we have $A^T A = I$, which gives us the following equations:

$$a^2 + c^2 = 1$$
$$ab + cd = 0$$
$$ab + cd = 0$$
$$b^2 + d^2 = 1$$

From the first and last equations, we can write:

$$c = \pm\sqrt{1 - a^2}, \quad d = \pm\sqrt{1 - b^2}$$

Now, let's consider the second equation, $ab + cd = 0$. We have four possible cases:

1. $c = \sqrt{1 - a^2}$ and $d = \sqrt{1 - b^2}$:

In this case, we have:

$$ab + \sqrt{1 - a^2}\sqrt{1 - b^2} = 0$$

Squaring both sides, we get:

$$a^2 b^2 + 2ab\sqrt{(1 - a^2)(1 - b^2)} + (1 - a^2)(1 - b^2) = 0$$

Since $a^2 + b^2 = 1$, we can simplify the equation to:

$$2ab\sqrt{(1-a^2)(1-b^2)} = -1$$

However, the left-hand side of the equation is non-negative, while the right-hand side is negative. This case is not possible.

2. $c = \sqrt{1-a^2}$ and $d = -\sqrt{1-b^2}$:

In this case, we have:

$$ab - \sqrt{1-a^2}\sqrt{1-b^2} = 0$$

Squaring both sides, we get:

$$a^2b^2 - 2ab\sqrt{(1-a^2)(1-b^2)} + (1-a^2)(1-b^2) = 0$$

Since $a^2 + b^2 = 1$, we can simplify the equation to:

$$2ab\sqrt{(1-a^2)(1-b^2)} = 1$$

Let $a = \cos\varphi$ and $b = \sin\varphi$. Then, we have:

$$c = \sqrt{1-a^2} = \sin\varphi, \quad d = -\sqrt{1-b^2} = -\cos\varphi$$

So, in this case, the matrix A can be expressed as:

$$A = \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix}$$

3. $c = -\sqrt{1-a^2}$ and $d = \sqrt{1-b^2}$:

In this case, we have:

$$ab + \sqrt{1-a^2}\sqrt{1-b^2} = 0$$

Squaring both sides, we get:

$$a^2b^2 + 2ab\sqrt{(1-a^2)(1-b^2)} + (1-a^2)(1-b^2) = 0$$

Since $a^2 + b^2 = 1$, we can simplify the equation to:

$$2ab\sqrt{(1-a^2)(1-b^2)} = -1$$

However, the left-hand side of the equation is non-negative, while the right-hand side is negative. This case is not possible.

4. $c = -\sqrt{1-a^2}$ and $d = -\sqrt{1-b^2}$:

In this case, we have:

$$ab - \sqrt{1-a^2}\sqrt{1-b^2} = 0$$

Squaring both sides, we get:

$$a^2b^2 - 2ab\sqrt{(1-a^2)(1-b^2)} + (1-a^2)(1-b^2) = 0$$

Since $a^2 + b^2 = 1$, we can simplify the equation to:

$$2ab\sqrt{(1-a^2)(1-b^2)} = 1$$

Let $a = \cos\varphi$ and $b = \sin\varphi$. Then, we have:

$$c = -\sqrt{1-a^2} = -\sin\varphi, \quad d = -\sqrt{1-b^2} = \cos\varphi$$

So, in this case, the matrix A can be expressed as:

$$A = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$$

In conclusion, every orthogonal 2×2 matrix can be expressed in either one of the two forms:

$$\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \text{ or } \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix}$$

Problem 6 (a) Let A be an invertible square matrix such that (λ, x) is an eigenpair, $\lambda \neq 0, x \neq 0$. Show that $(1/\lambda, x)$ is an eigenpair of A^{-1}. (b) Let the matrix A have exactly two eigenvalues $\lambda_1 = 1, \lambda_2 = 1/2$, and the corresponding eigenvectors $v_1 = (1,1)^T, v_2 = (-1,1)^T$. Find the limit $\lim_{k\to\infty} A^k$.

Problem 6 solution

(a) Let A be an invertible square matrix such that (λ, x) is an eigenpair, $\lambda \neq 0$, $x \neq 0$. This means that:

$$Ax = \lambda x$$

Since A is invertible, we can multiply both sides of the equation by A^{-1}:

$$A^{-1}Ax = A^{-1}\lambda x$$

Since $A^{-1}A = I$, where I is the identity matrix, we have:

$$Ix = \frac{1}{\lambda}A^{-1}x$$

This implies that:

$$A^{-1}x = \frac{1}{\lambda}x$$

So, $(1/\lambda, x)$ is an eigenpair of A^{-1}.

(b) Let the matrix A have exactly two eigenvalues $\lambda_1 = 1$, $\lambda_2 = 1/2$, and the corresponding eigenvectors $v_1 = (1,1)^T$, $v_2 = (-1,1)^T$. We can write any vector x as a linear combination of the eigenvectors:

$$x = c_1v_1 + c_2v_2$$

Now, let's find the k-th power of A:

$$A^kx = A^k(c_1v_1 + c_2v_2) = c_1A^kv_1 + c_2A^kv_2$$

Since v_1 and v_2 are eigenvectors of A, we have:

$$A^kv_1 = \lambda_1^kv_1 = v_1$$

$$A^kv_2 = \lambda_2^kv_2 = \left(\frac{1}{2}\right)^k v_2$$

So, we can write:

$$A^kx = c_1v_1 + c_2\left(\frac{1}{2}\right)^k v_2$$

Now, let's find the limit as $k \to \infty$:

$$\lim_{k\to\infty} A^k x = c_1 v_1 + c_2 \lim_{k\to\infty} \left(\frac{1}{2}\right)^k v_2 = c_1 v_1$$

Since the limit of $A^k x$ as $k \to \infty$ is a multiple of the eigenvector v_1, the limit of the matrix A^k as $k \to \infty$ is a matrix that maps every vector to a multiple of v_1. This matrix can be represented as:

$$\lim_{k\to\infty} A^k = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$