PHYS-C0252 — Quantum Mechanics

Final examination

Thursday June 8th 2023, 13-16

Instructions

- Your answers should be legible and appropriately numbered.
- Your answers should contain relevant intermediate steps and explanations for the calculations.
- No calculators, cheat sheets or other materials are allowed.
- Return the exam paper when you are finished.

1. Briefly define the following terms in the context of quantum mechanics:

   (a) (*1 point*) Qubit

   (b) (*1 point*) Quantum state of a physical system

   (c) (*1 point*) Quantum measurement

   (d) (*1 point*) Degenerate eigenvalues

   (e) (*1 point*) Density operator

   (f) (*1 point*) Heisenberg uncertainty principle

2. A quantum system described by a Hamiltonian $\hat{H}$ is in the state

$$|\psi\rangle = N\left[\frac{1}{\sqrt{2}}|\phi_1\rangle - \frac{\mathrm{i}}{\sqrt{2}}|\phi_2\rangle + \frac{1}{\sqrt{5}}(1+2\mathrm{i})\,|\phi_3\rangle + \sqrt{5}|\phi_4\rangle\right],$$

   where $|\phi_n\rangle$ are the eigenstates of energy such that $\hat{H}|\phi_n\rangle = nE_0|\phi_n\rangle$, E_0 has units of energy, and $N \in \mathbb{R}$.

   (a) (*2 points*) Find a suitable scalar N such that $|\psi\rangle$ is normalized.

   (b) (*2 points*) Let the energy of $|\psi\rangle$ be measured. Give all possible measurement results and their corresponding probabilities. Assume that the measurement is ideal, i.e., no measurement errors occur.

   (c) (*2 points*) Consider an operator $\hat{X}$, the action of which on $|\phi_n\rangle$ ($n = 1, 2, 3, 4$) is defined by $\hat{X}|\phi_n\rangle = (n+5)x_0|\phi_n\rangle$, where x_0 is a real-valued scalar. Suppose that a measurement of the energy of the above-defined $|\psi\rangle$ yields $3E_0$. Assume that immediately afterwards, we ideally measure the physical quantity corresponding to $\hat{X}$. What is the value for the quantity obtained in the latter measurement?

3. Consider a classical LC oscillator, consisting of a parallel connection of an inductor with inductance L and a capacitor with capacitance C. The circuit can act as an electrical resonator, storing energy oscillating at the resonance frequency

$$\omega = \frac{1}{\sqrt{LC}}.$$

From classical circuit theory we know that the energy of the LC oscillator is

$$E = \frac{1}{2}LI^2 + \frac{Q^2}{2C},$$

where I is the current in the circuit and Q is the electrical charge stored in the capacitor. Using Q as a generalized coordinate and noting that $I = \dot{Q}$, the corresponding conjugate momentum can be written as $\Phi = L\dot{Q} = LI$. By canonical quantization $Q \to \hat{Q}$, $\Phi \to \hat{\Phi}$, where $[\hat{\Phi}, \hat{Q}] = -\mathrm{i}\hbar$, we can then write the quantized Hamiltonian as

$$\hat{H} = \frac{\hat{\Phi}^2}{2L} + \frac{\hat{Q}^2}{2C}.$$

(a) (*3 points*) The quantized Hamiltonian can be diagonalized using the following linear transformation for the generalized momentum $\hat{\Phi}$ and the generalized coordinate $\hat{Q}$:

$$\hat{\Phi} = \sqrt{\frac{\hbar\omega L}{2}}(\hat{a} + \hat{a}^\dagger),$$
$$\hat{Q} = \mathrm{i}\sqrt{\frac{\hbar\omega C}{2}}(\hat{a} - \hat{a}^\dagger),$$

where the transformation defines the operator $\hat{a}$. Show that the diagonalized Hamiltonian can be written as

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right). \tag{1}$$

(b) (*3 points*) The eigenstates $\{|n\rangle\}_{n=0}^{\infty}$ of the Hamiltonian (1) satisfy

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle$$
$$\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\,.$$

Suppose the system is in the state $|\psi\rangle = |2\rangle$. Compute the magnitude of charge fluctuations, $\Delta Q = \sqrt{\langle\hat{Q}^2\rangle - \langle\hat{Q}\rangle^2}$ in the circuit.

4. Recall the Bloch sphere representation of quantum states: A general state of a two-dimensional quantum system

$$|\psi\rangle = \cos(\theta/2)\,|0\rangle + \mathrm{e}^{\mathrm{i}\phi}\sin(\theta/2)\,|1\rangle\,,$$

can be mapped onto the unit sphere using the Bloch vector

$$\psi_{\mathrm{B}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta),$$

where $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$ are the polar and azimuthal angles, respectively.

(a) (*2 points*) Write the quantum states at the poles of the Bloch sphere, i.e., $\psi_{\mathrm{B}} = (0, 0, -1)$ and $\psi_{\mathrm{B}} = (0, 0, 1)$, in the form $|\psi\rangle = a|0\rangle + b|1\rangle$, where $a, b \in \mathbb{C}$.

(b) (*2 points*) Consider an operator $\hat{G}(\gamma)$ which maps a Bloch vector as

$$(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \rightarrow (\sin\theta\cos(\phi+\gamma), \sin\theta\sin(\phi+\gamma), \cos\theta),$$

where $\gamma \in \mathbb{R}$. Show that the action of this operator on a ket vector can be represented as the matrix

$$G(\gamma) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\gamma} \end{bmatrix},$$

in the basis $\{|0\rangle, |1\rangle\}$.

(c) (*2 points*) Suppose a qubit is initially in the state $|\psi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Let us first apply the operator $\hat{G}(\pi/2)$, and then the operator

$$\hat{R}_x(\pi/2) \mathrel{\hat{=}} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & 1 \end{bmatrix},$$

to the qubit. Write the Bloch vector of the final state $|\psi_{\mathrm{f}}\rangle = \hat{R}_x(\pi/2)\hat{G}(\pi/2)|\psi_0\rangle$ of the qubit in the form $|\psi_{\mathrm{f}}\rangle = a|0\rangle + b|1\rangle$, where $a, b \in \mathbb{C}$.

5. A particle is located inside a one-dimensional infinite potential well with hard walls at $x = 0$ and $x = a$, which can be described by the potential

$$V(x) = \begin{cases} 0, & \text{if } 0 \le x \le a \\ \infty, & \text{otherwise} \end{cases}.$$

(a) (*2 points*) Write the Hamiltonian of this system in the position representation and show that

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right),$$
$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2},$$

for $n \in \{1, 2, 3, \ldots\}$ are the eigenfunctions and the corresponding eigenvalues of the Hamiltonian that satisfy the appropriate boundary conditions. You can assume that the above wave functions are normalized correctly (no need to show this).

(b) (*4 points*) Suppose that at $t = 0$, the system is prepared in the state

$$\psi(x, t=0) = \frac{1}{\sqrt{2}}[\psi_1(x) + \psi_2(x)].$$

Using the Schrödinger equation, find $\psi(x,t)$ for arbitrary t, and determine the expectation value of the position of the particle, $\langle x(t)\rangle_\psi = \langle\psi(t)|\hat{x}|\psi(t)\rangle$. You may use some or all of the formulas given below

$$\sin^2(\xi) = \frac{1}{2}\left[1 - \cos(2\xi)\right]$$

$$\int x\sin(\alpha x)\sin(2\alpha x)\mathrm{d}x = \frac{12\alpha x\sin^3(\alpha x) + 9\cos(\alpha x) - \cos(3\alpha x)}{18\alpha^2} + C$$

$$\langle x(t)\rangle_{\psi_1} = \langle x(t)\rangle_{\psi_2} = \frac{1}{2}a,$$

where C is an integration constant and $\langle x(t)\rangle_{\psi_i} = \langle\psi_i(t)|\hat{x}|\psi_i(t)\rangle$.

Bonus: (*0.5 points*) How long did it take for you to finish the exam?