Aalto University
Department of Mathematics and Systems Analysis
MS-E1200 - Lie groups and Lie algebras

Exam T01 / KT
Thu 2.6. at 13–16
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Allowed equipment:

- Writing equipment.
- A handwritten **memory aid sheet**. The memory aid sheet must be of size A4 with text only on one side, and it must contain your name and student number in the upper right corner.

The exam consists of 4 problems, each worth 6 points. Up to 6 bonus points from exercises are added to the exam score.

Problem 1. Let $n \geq 2$. We consider it known that

$$\mathrm{SL}_n(\mathbb{C}) = \{g \in \mathrm{GL}_n(\mathbb{C}) \mid \det(g) = 1\},$$
$$\mathrm{O}_n = \{g \in \mathrm{GL}_n(\mathbb{R}) \mid X^\top X = \mathbb{I}\},$$

are matrix Lie groups.

(a) Prove that one of the above matrix Lie groups is compact. **(2 pt)**
(b) Prove that the other one of the above is not compact. **(2 pt)**
(c) Prove that one of the above matrix Lie groups is not connected. **(2 pt)**

Problem 2. Let $\mathfrak{g}$ be a Lie algebra of compact type, i.e., a finite-dimensional real Lie algebra, which has an inner product

$$\mathsf{B}(\cdot,\cdot)\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R} \qquad \text{(symmetric, positive-definite, bilinear)}$$

which is ad-invariant in the sense that

$$\mathsf{B}\big([Z,X],Y\big) + \mathsf{B}\big(X,[Z,Y]\big) = 0 \qquad \text{for all } X,Y,Z \in \mathfrak{g}.$$

Let $Z_1, \ldots, Z_d$ be an orthonormal basis of $\mathfrak{g}$ with respect to this invariant inner product, i.e., $\mathsf{B}(Z_i, Z_j) = \delta_{ij}$ for $i,j \in \{1,\ldots,d\}$.

(a) Expanding the Lie brackets in the orthonormal basis, for all $i,j \in \{1,\ldots,d\}$ we can write $[Z_i, Z_j] = \sum_{k=1}^d b_{ij}^k Z_k$ with certain coefficients $b_{ij}^k \in \mathbb{R}$. Prove that these coefficients satisfy the following: **(2 pt)**

$$b_{ij}^k = -b_{ji}^k \qquad \text{and} \qquad b_{ij}^k = -b_{ik}^j \qquad \text{for all } i,j,k \in \{1,\ldots,d\}.$$

(b) Let $\vartheta\colon \mathfrak{g} \to \mathrm{End}(V)$ be a representation of $\mathfrak{g}$, and for each $j \in \{1,\ldots,d\}$ denote by $\mathcal{Z}_j := \vartheta(Z_j)$ the corresponding linear map $V \to V$. Define the linear map $\mathcal{Q} = \sum_{j=1}^d \mathcal{Z}_j^2$. Using the results of part (a), prove that for each $i \in \{1,\ldots,d\}$ we have $\mathcal{Z}_i\mathcal{Q} = \mathcal{Q}\mathcal{Z}_i$. **(2 pt)**
(c) Suppose that $\vartheta\colon \mathfrak{g} \to \mathrm{End}(V)$ is an irreducible representation of $\mathfrak{g}$ in a finite-dimensional $\mathbb{C}$-vector space V. Let $\mathcal{Q}\colon V \to V$ be defined as in part (b). Prove that there exists a constant $c \in \mathbb{C}$ such that $\mathcal{Q} = c\,\mathrm{id}_V$. **(2 pt)**

Problem 3. Consider $G = \{g \in \mathrm{GL}_4(\mathbb{R}) \mid g^\top J g = J\}$, where

$$J = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

(a) Show that $G \subset \mathrm{GL}_4(\mathbb{R})$ is a closed subgroup. **(3 pt)**
(This shows that G is a matrix Lie group.)

(b) The Lie algebra of G is defined as $\mathfrak{g} = \{X \in \mathbb{R}^{4\times 4} \mid \forall t \in \mathbb{R} : e^{tX} \in G\}$. Prove directly from this definition that $\mathfrak{g}$ consists of exactly those matrices $X \in \mathbb{R}^{4\times 4}$ which satisfy $X^\top = JXJ$. **(2 pt)**

(c) Using part (b), calculate the dimension of $\mathfrak{g}$. **(1 pt)**

Problem 4. Consider the symmetric group on three letters, i.e., the group $\mathfrak{S}_3$ of bijective functions $\sigma\colon \{1,2,3\} \to \{1,2,3\}$. The conjugacy classes of $\mathfrak{S}_3$ are $\{e\}$ (neutral element), $\{(12),(13),(23)\}$ (transpositions), and $\{(123),(132)\}$ (three-cycles).

The group $\mathfrak{S}_3$ has two one-dimensional complex representations: the trivial representation given by $\sigma \mapsto \mathrm{id}_{\mathbb{C}}$ and the alternating representation given by $\sigma \mapsto \mathrm{sgn}(\sigma)\,\mathrm{id}_{\mathbb{C}}$, for $\sigma \in \mathfrak{S}_3$.

Let $P = \mathbb{C}[x_1, x_2, x_3]$ denote the space of polynomials in three indeterminates x_1, x_2, x_3 with complex coefficients.

(a) For $p \in P$ and $\sigma \in \mathfrak{S}_3$, let $\sigma.p$ denote the polynomial

$$(\sigma.p)(x_1, x_2, x_3) = p\big(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}\big).$$

Show that this (more precisely the formula $\sigma \mapsto (p \mapsto \sigma.p)$) defines a representation of $\mathfrak{S}_3$ on the vector space P. **(2 pt)**

For $d \in \mathbb{N}$, let $P_d \subset P$ denote the subspace of homogeneous polynomials of degree d. Clearly $P_d \subset P$ is a subrepresentation of P.

(b) The space P_1 of degree 1 homogeneous polynomials has a basis x_1, x_2, x_3. Calculate the character of the representation P_1 of $\mathfrak{S}_3$. Find the multiplicities of the trivial and alternating representation in P_1. Conclude that P_1 contains also a two-dimensional irreducible subrepresentation. **(2 pt)**

(c) The space P_3 of degree 3 homogeneous polynomials has a basis

$$x_1^3,\ x_1^2x_2,\ x_1^2x_3,\ x_1x_2^2,\ x_1x_2x_3,\ x_1x_3^2,\ x_2^3,\ x_2^2x_3,\ x_2x_3^2,\ x_3^3.$$

Find a non-zero polynomial $q \in P_3$ such that the one-dimensional vector space $\mathbb{C}q \subset P_3$ spanned by it is a subrepresentation isomorphic to the alternating representation of $\mathfrak{S}_3$. **(2 pt)**