Aalto University
Department of Mathematics and Systems Analysis
MS-E1200 - Lie groups and Lie algebras

Exam T01 / KT
Thu 8.6. at 13–16
K Kytölä & D Adame-Carrillo

Allowed equipment:

- Writing equipment.
- A handwritten **memory aid sheet**. The memory aid sheet must be of size A4 with text only on one side, and it must contain your name and student number in the upper right corner.

The exam consists of 4 problems, each worth 6 points.

Problem 1. Consider the following set

$$G := \left\{ \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}.$$

(a) Show that G is a subgroup of the group $\mathrm{GL}_3(\mathbb{R})$ of invertible 3×3 real matrices. **(3 pts)**
(b) Show that $G \subset \mathrm{GL}_3(\mathbb{R})$ is a closed subset. **(1 pts)**
(c) Is G compact? **(1 pts)**
(d) Is G path-connected? **(1 pts)**
Recall: Path-connected means: for all $M, M' \in G$, there exists a continuous $\gamma \colon [0,1] \to G$ such that $\gamma(0) = M$ and $\gamma(1) = M'$.

Problem 2. Let $n \in \mathbb{N}$. Consider the special unitary group

$$\mathrm{SU}_n = \left\{ M \in \mathbb{C}^{n \times n} \;\middle|\; M^\dagger M = \mathbb{I} \text{ and } \det(M) = 1 \right\}.$$

(a) Show that SU_n is a closed subgroup of the group $\mathrm{GL}_n(\mathbb{C})$ of invertible $n \times n$ complex matrices. **(2 pts)**
(b) Denoting by $\exp \colon \mathbb{C}^{n \times n} \to \mathrm{GL}_n(\mathbb{C})$ the matrix exponential, define

$$\mathfrak{su}_n = \left\{ X \in \mathbb{C}^{n \times n} \;\middle|\; \forall t \in \mathbb{R} : \exp(tX) \in \mathrm{SU}_n \right\}.$$

Show that $\mathfrak{su}_n$ consists of those $n \times n$ complex matrices X which satisfy $X^\dagger = -X$ and $\mathrm{tr}(X) = 0$. **(2 pts)**
(c) Using the characterization of $\mathfrak{su}_n$ in part (b), show that $\mathfrak{su}_n$ is an $\mathbb{R}$-vector subpace of the space $\mathbb{C}^{n \times n}$ of all complex $n \times n$ matrices. Compute its dimension $\dim(\mathfrak{su}_n)$. **(2 pts)**

Problem 3. Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{a}$ a real vector space. Consider their vector space direct sum

$$\mathfrak{h} := \mathfrak{g} \oplus \mathfrak{a} = \left\{ (X, A) \;\middle|\; X \in \mathfrak{g},\, A \in \mathfrak{a} \right\}.$$

(a) Suppose that $c\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{a}$ is a bilinear function which satisfies

$$c(X,Y) = -c(Y,X) \text{ and } c(X,[Y,Z]) + c(Y,[Z,X]) + c(Z,[X,Y]) = 0$$

for all $X, Y, Z \in \mathfrak{g}$. Show that the mapping $[\cdot,\cdot]_c\colon \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}$ defined by

$$[(X,A),(Y,B)]_c = ([X,Y], c(X,Y))$$

is a Lie bracket on the vector space $\mathfrak{h}$. **(2 pts)**

(b) Suppose that $c\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{a}$ is as in (a), and $b\colon \mathfrak{g} \to \mathfrak{a}$ is a linear map. Show that $c'\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{a}$ given by $c'(X,Y) = c(X,Y) + b([X,Y])$, for $X, Y \in \mathfrak{g}$, is bilinear and for all $X, Y, Z \in \mathfrak{g}$ we have **(2 pts)**

$$c'(X,Y) = -c'(Y,X) \text{ and } c'(X,[Y,Z]) + c'(Y,[Z,X]) + c'(Z,[X,Y]) = 0.$$

(c) Suppose that $c\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{a}$ and $b\colon \mathfrak{g} \to \mathfrak{a}$ and $c'\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{a}$ are as in (a) and (b) above. Let $\mathfrak{h}$ and $\mathfrak{h}'$, respectively, be the Lie algebras obtained by equipping the same vector space $\mathfrak{g} \oplus \mathfrak{a}$ with the Lie brackets $[\cdot,\cdot]_c$ and $[\cdot,\cdot]_{c'}$ associated to c and c', respectively, by the prescription of part (a). Show that $\mathfrak{h}$ and $\mathfrak{h}'$ are isomorphic as Lie algebras. **(2 pts)**

Problem 4. An *irreducible fundamental system* of rank 2 is a basis α, β of the Euclidean plane $\mathbb{R}^2$ (with the usual inner product denoted by $\langle \cdot, \cdot \rangle$) such that $2\frac{\langle \alpha,\beta \rangle}{\langle \beta,\beta \rangle}, 2\frac{\langle \alpha,\beta \rangle}{\langle \alpha,\alpha \rangle} \in \{-1, -2, -3\}$ and at least one of the numbers $2\frac{\langle \alpha,\beta \rangle}{\langle \beta,\beta \rangle}$ and $2\frac{\langle \alpha,\beta \rangle}{\langle \alpha,\alpha \rangle}$ is equal to -1.

(a) By rotations, reflections, scaling, and relabeling the basis vectors, we may assume $\alpha = (1,0)$, $\beta = (x,y)$ with $y > 0$, and $2\frac{\langle \alpha,\beta \rangle}{\langle \alpha,\alpha \rangle} = -1$. In this case, find all possible vectors β such that α, β is an irreducible fundamental system. **(3 pts)**

(b) Define the reflections $\sigma, \tau\colon \mathbb{R}^2 \to \mathbb{R}^2$ of the plane by $\sigma(v) := v - 2\frac{\langle \alpha,v \rangle}{\langle \alpha,\alpha \rangle}\alpha$ and $\tau(v) := v - 2\frac{\langle \beta,v \rangle}{\langle \beta,\beta \rangle}\beta$. For each of the irreducible fundamental systems in part (a), find the smallest set of points $\Phi \subset \mathbb{R}^2$ such that $\alpha, \beta \in \Phi$, and $\sigma(\phi), \tau(\phi) \in \Phi$ for all $\phi \in \Phi$. **(3 pts)**

Remark: One can use the results of this problem to obtain a classification of simple Lie algebras of rank 2.