Mathematics
Aalto University

Hannukainen / Autio

MS-E1653 Finite Element Method

Exam 18.4.2023

Please fill in clearly *on every sheet* the data on you and the examination. On *Examination code* mark the course code, title and text mid-term or final examination.

You have two options

1. Solve all problems. Grade is based only on the exam.
2. Solve any three problems. Grade is based on exercise points + exam points. To choose this option, you must have completed the final project.

The exam time is three hours (3h). No electronic calculators or materials are allowed.

1. Let $c \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^2$ be a domain with a sufficiently nice boundary. Consider the strong problem: find u such that

$$\begin{cases} -\Delta u + cu = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{1}$$

   (a) Derive the weak form of the strong problem (1)

   (b) Assume that $c > 0$. Use the Lax-Milgram lemma to show that there exists a unique solution to the weak problem.

   (c) For which $c \in \mathbb{R}$ can Lax-Milgram lemma guarantee existence of a unique solution ?

   Hint: Denote the optimal constant in Poincare-Friedrichs inequality by C_Ω.

2. Let the reference element be $\hat{K} = (0,0), (1,0), (0,1)$ and the reference basis functions

$$\hat{\varphi}_1(\hat{x},\hat{y}) = 1 - \hat{x} - \hat{y} \quad \hat{\varphi}_2(\hat{x},\hat{y}) = \hat{x} \quad \text{and} \quad \hat{\varphi}_3(\hat{x},\hat{y}) = \hat{y}$$

   Consider the element $K = (\frac{1}{2}, 0), (\frac{3}{2}, 2), (\frac{1}{2}, 1)$.

   (a) Compute the affine mapping from $\hat{K}$ to K

   (b) Give the definition of first order nodal based functions on element K. How are these functions related to the reference basis functions ?

   (c) Compute the gradient of each of the global basis functions on K

   (d) Consider the bilinear form $(\nabla u, \nabla v)$. Compute the contribution of the element K to the system matrix.

   Hint: Inverse of any 2×2-matrix can be computed as

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}. \tag{2}$$

3. Let $a : V \times V \to \mathbb{R}$ be a bilinear form and $L : V \to \mathbb{R}$ a linear functional. In addition, assume that b is bounded and a is continuous as well as elliptic. Consider the problem : find $u \in V$ such that

$$a(u, v) = L(v) \quad \forall v \in V. \tag{3}$$

Let $V_h \subset V$. The finite element approximation to problem (2) is : find $u_h \in V_h$ such that

$$a(u_h, v_h) = L(v_h) \quad \forall v_H \in V_h.$$

(a) Formulate and prove Cea's Lemma

(b) Explain how Cea's lemma can be used to derive finite element error estimates.

4. Let V_h be the first order FE-space related to partition $\{x_j\}_{j=1}^N$ of I. Assume there exists ρ independent of N and h such that

$$\min_{i \in \{1,\ldots,N-1\}} (x_{i+1} - x_i) \geq \rho h.$$

Prove *the inverse inequality*: there exists a constant C dependent on ρ but independent of v_h and h such that

$$\|v_h'\|_{L^2(I)} \leq C h^{-1} \|v_h\|_{L^2(I)} \quad \text{for all } v_h \in V_h. \tag{4}$$

Use the scaling argument and the following result :

$$\|\hat{p}'\|_{L^2(0,1)} \leq \widehat{C} \|\hat{p}\|_{L^2(0,1)} \quad \text{for all } \hat{p} \in P^1(0,1),$$

where constant $\widehat{C}$ is independent of $\hat{p}$.

Scaling Argument Let $a < b$, $k \in \{0, 1, \ldots\}$, $h_I := (b - a)$, and $r : (0,1) \mapsto (a,b)$ be defined as $r(\hat{t}) := (b-a)\hat{t} + a$. In addition, let $v \in H^k(a,b)$ and $\hat{v} \in H^k(0,1)$ be defined as $\hat{v}(\hat{t}) := v(r(\hat{t}))$. Then there holds that

$$\left\| \frac{d^{(k)} v}{dt^{(k)}} \right\|_{L^2(a,b)} = h_I^{(1-2k)/2} \left\| \frac{d^{(k)} \hat{v}}{d\hat{t}^{(k)}} \right\|_{L^2(0,1)}. \tag{5}$$

Here $\frac{d^{(0)} v}{dt^{(0)}} = v$ and $\frac{d^{(0)} \hat{v}}{d\hat{t}^{(0)}} = \hat{v}$.