Aalto University
Department of Mathematics and Systems Analysis
MS-C1541 — Metric spaces, 2022-2023/III

Exam T0 / KT
23.2.2023
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Allowed equipment:

- *writing equipment,*
- *memory aid sheet (A4, hand-written, one-sided, w/ name + student number).*

This is the exam sheet for both the final exam (T0) and the course exam (KT) of MS-C1541 Metric spaces. The grading is based on either

- 100% final exam (T0);
- 60% course exam (KT) + 40% exercises (during the period III course).

You can attempt both options, and the one leading to the more favorable grade is taken into account.

Depending on the option above, you should solve the following problems:

- **Final exam (T0):** Solve all five problems.
- **Course exam (KT):** Choose any four of the five problems.

(If you solve all problems, the best four are taken into consideration for the course completion option based on course exam + exercises.)

PROBLEMS

Problem 1.

Express in two different but logically equivalent ways what it means for...

(a) ... a function $f\colon X \to Y$ between two metric spaces (X, d_X) and (Y, d_Y) to be continuous; **(2 pts)**

(b) ... a subset $A \subset X$ in a metric space (X, d) to be closed; **(2 pts)**

(c) ... a metric space (X, d) to be compact. **(2 pts)**

<u>Hint</u>: *As one of the equivalent ways, you can use the definition of the property in question, and as the other you can use some known characterization of it.*

Problem 2.

Possible or not? In each subproblem below, give an example of the specified kind, or explain why such an example cannot exist. In your justifications for the requested properties or their impossibility you can (and should) use results from the course.

(a) A subset $A \subset X$ in a metric space (X, d) such that A is neither open nor closed in X. **(3 pts)**

(b) A continuous function $f\colon X \to Y$ between two metric spaces (X, d_X) and (Y, d_Y), and a bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X, such that the sequence $\big(f(x_n)\big)_{n\in\mathbb{N}}$ is not bounded. **(3 pts)**

Problem 3.
Let $\|\cdot\|_2$ denote the Euclidean norm on the three-dimensional vector space $\mathbb{R}^3$ and let $\|\cdot\|_1$ denote the ℓ^1-norm on $\mathbb{R}^3$. For $\vec{v} = (x, y, z) \in \mathbb{R}^3$ these are given by

$$\|\vec{v}\|_2 = \sqrt{x^2 + y^2 + z^2}, \qquad \text{and} \qquad \|\vec{v}\|_1 = |x| + |y| + |z|.$$

(a) Prove that there exists a constant $A > 0$ such that for any $\vec{v} \in \mathbb{R}^3$ we have $\|\vec{v}\|_2 \leq A\,\|\vec{v}\|_1$. **(2 pts)**
<u>Hint</u>: *It is possible to prove this without explicitly specifying the constant A, but you may also prove this directly with $A = 1$.*

(b) Prove that there exists a constant $B > 0$ such that for any $\vec{v} \in \mathbb{R}^d$ 3 we have $\|\vec{v}\|_1 \leq B\,\|\vec{v}\|_2$. **(2 pts)**
<u>Hint</u>: *It is possible to prove this without explicitly specifying the constant B, but you may also prove this directly with $B = \sqrt{3}$. The Cauchy-Schwarz inequality may be helpful.*

(c) Show that in parts (a) and (b), it would be impossible to use constants $A < 1$ and $B < \sqrt{3}$ (as the inequalities are required to hold for all $\vec{v} \in \mathbb{R}^d$). 3 **(2 pts)**
<u>Hint</u>: *Judiciously chosen explicit vectors provide counterexamples.*

Problem 4.
Consider the space X consisting of all real-valued functions on the real line $\mathbb{R}$. Show that the formula

$$\mathsf{d}(f, g) = \sup\left\{ \frac{|f(t) - g(t)|}{1 + |f(t) - g(t)|} \;\middle|\; t \in \mathbb{R} \right\} \qquad \text{for } f, g \in X$$

defines a metric on X. **(6 pts)**
Remark: This metric induces the topology of uniform convergence of functions.

Problem 5.
Consider the space

$$C([0,\pi]) = \Big\{ f\colon [0,\pi] \to \mathbb{R} \;\Big|\; f \text{ is continuous} \Big\}$$

of continuous real-valued functions on the closed interval $[0,\pi]$, equipped with the supremum norm $\|f\|_\infty = \sup_{t\in[0,\pi]} |f(t)|$ for $f \in C([0,\pi])$. Define a function $F\colon C([0,\pi]) \to C([0,\pi])$ so that its value at $f \in C([0,\pi])$ is the function $F(f) \in C([0,\pi])$ given by

$$[F(f)](t) \;=\; \pi + \int_0^t e^{-s} \cos\left(\frac{f(s)}{4}\right) \mathrm{d}s \qquad \text{for} \qquad t \in [0,\pi].$$

(a) Show that $F\colon C([0,\pi]) \to C([0,\pi])$ is M-Lipschitz with $M = \frac{\pi}{4}$. **(2 pts)**
<u>Hint</u>: *You may use the fact that $|\cos(x) - \cos(y)| \leq |x - y|$ for any $x, y \in \mathbb{R}$.*

(b) Show that there exists a unique $g \in C([0,\pi])$ such that $F(g) = g$. **(3 pts)**
<u>Hint</u>: *Results from the course can be used.*

(c) Show that the function g of part (b) satisfies the differential equation

$$g'(t) = e^{-t} \cos\left(\frac{g(t)}{4}\right) \qquad \text{for all} \qquad t \in (0,\pi)$$

with initial condition $g(0) = \pi$. **(1 pt)**